Jacobian Conjecture:
A Birational Geometry Approach

Alexander Borisov

University of Pittsburgh

March 5, 2013

Johns Hopkins University

Algebraic Geometry and Number Theory Seminar

Suppose $n \in \mathbb{N}$. Suppose

$$F_1(x_1, x_2, ..., x_n), F_2(x_1, x_2, ..., x_n), ..., F_n(x_1, x_2, ..., x_n)$$

are n polynomials in n variables with complex coefficients. Together they define a polynomial self-map F of the n-dimensional affine space $\mathbb{C}^n$.

The Jacobian of F is the determinant of the matrix

$$\left(\frac{\partial F_i}{\partial x_j}\right)_{i,j=1...n}$$

Implicit Function Theorem: F is locally invertible at $(x_1, ...x_n)$ iff $J(F)(x_1, ..., x_n) \neq 0$.

$J(F)$ is a polynomial in $x_1, x_2, ...x_n$. Every non-constant complex polynomial has complex roots. Therefore the map F is everywhere locally one-to-one if and only if $J(F)$ is a non-zero constant.

Jacobian Conjecture (O. Keller, 1939)

$J(F)$ is a non-zero constant if and only if F is invertible.

Remarks

1) The "if" part of the Conjecture follows from the discussion above.

2) Keller's original question was about polynomials with integer coefficients.

3) The name "Jacobian Conjecture" was coined by S. Abhyankar in the 1970s.

Simple Observations

$n = 1$: $f'(x)$ is a nonzero constant $\Leftrightarrow$ f is linear.

$J(F) \not\equiv 0 \Rightarrow F(\mathbb{C}^n)$ is dense in $\mathbb{C}^n$. The field $\mathbb{C}(x_1, ..., x_n)$ is a *finite* extension of $\mathbb{C}(F_1, ..., F_n)$. Its degree is the number of preimages of a generic point.

Theorem(Keller, 1939)

If $\mathbb{C}(x_1, ..., x_n) = \mathbb{C}(F_1, ..., F_n)$, then

$\mathbb{C}[x_1, ..., x_n] = \mathbb{C}[F_1, ..., F_n]$

In particular, if F is injective, then it is surjective and the inverse map is also polynomial.

The First "Proof" of the JC

The map F is a finite unramified covering of $A^n(\mathbb{C})$. Because $A^n(\mathbb{C})$ is simply connected, its degree must be 1.

Mistake: The map is quasi-finite, but not necessarily finite. In other words, some points in $\mathbb{C}^n$ may have fewer preimages than their neighbors.

Example. $F(x,y) = (x(xy-1), y(xy+1))$.

For generic (a,b), $F(x,y) = (a,b)$ has three solutions. But for $(a,b) = (0,0)$, there is only one solution: $(x,y) = (0,0)$. This is not a local issue: $J(F)(0,0) = -1$. Also, points with $a = 0, b \neq 0$ or $a \neq 0, b = 0$ have two preimages. When approaching such points by a curve, some preimages go to infinity.

Main Approaches and Results

K-theoretic approach. Starting from a counter-example, new counter-examples are created, with smaller degree polynomials in more variables.

Theorem(H. Bass, E. Connell, D. Wright, 1982; Yagzhev, 1980) The Jacobian Conjecture is true for all n if it is true for polynomials of degree at most three.

Remarks

1) In 1983 Druzkowski improved on this result by restricting to $F_i = x_i + H_i^3$, where H_i are linear functionals on $\mathbb{C}^n$.

2) The Jacobian Conjecture is true for polynomials of degree at most two (Wang, 1980).

Dimension 2 Results

The two-dimensional Jacobian Conjecture received a lot of extra attention for three reasons:

1) It is a deceivingly simple question about two polynomials in two variables.

2) It is much more likely to be true than the full Jacobian Conjecture.

3) Geometry of complex surfaces is much easier than the geometry of higher-dimensional complex varieties. In particular, polynomial automorphisms of surfaces are well understood (they are all "tame"). In higher dimensions there are "wild" automorphisms.

Magnus, 1955. JC is true for polynomials of coprime degrees. Improved by Nagata to gcd at most 8 (in 1989).

Abhyankar, Tata Lecture Notes, 1977. The JC is reduced to the polynomials with very restricted Newton polygons.

Abhyankar also proved the JC in the case when the field extension is Galois.

Related result:

Abhyankar-Moh-Suzuki Theorem.

(Abhyankar-Moh, 1973; Suzuki, 1974) Any algebraic embedding of $A^1(\mathbb{C})$ into $A^2(\mathbb{C})$ is equivalent to the embedding of a coordinate line.

Moh, 1983. JC is true for polynomials of degree less than or equal to 100.

Miyanishi, 2005. There exist unramified self-maps of algebraic surfaces that are quasi-finite but not finite.

Vitushkin, 1975. Constructed 3-sheeted non-algebraic counterexample.

Yu. Orevkov, 1986. The JC is true for maps of degree at most three (topological degree, not degrees of polynomials).

Improved by Domrina-Orevkov (1998) and Domrina (2000) to maps of degree up to four; Żoladek (2008) to degree five.

My Approach to the Two-dimensional JC

Suppose a counter-example exists. It gives a rational map from P^2 to P^2. After a sequence of blow-ups of points, we can get a surface X with two maps: $\pi : X \to P^2$ and $\phi : X \to P^2$:

$$\begin{tikzcd} & X \arrow[dl, "\pi"'] \arrow[dr, "\phi"] & \\ P^2 \arrow[rr, dashed] & & P^2 \end{tikzcd}$$

Note that X contains a Zariski open subset isomorphic to A^2 and its complement, $\pi^*((\infty))$, is a tree of smooth rational curves. We will call these curves exceptional, or curves at infinity. The structure of this tree is easy to understand inductively, as it is built from a single curve (∞) on P^2 by a sequence of two operations: blowing up a point on one of the curves or blowing up a point of intersection of two curves. One can keep track of the intersection pairing.

The Difficulties:

1) A non-inductive description does not exist.

2) The exceptional curves on X may behave very differently with respect to the map ϕ.

There are four types of curves E.

type 1) $\phi(E) = (\infty)$

type 2) $\phi(E)$ is a point on (∞)

type 3) $\phi(E)$ is a curve primarily on A^2.

type 4) $\phi(E)$ is a point on A^2.

3) The self-intersections of curves change under subsequent blow-ups.

What a mess!

First Idea: New labels

If $X \setminus A^2 = \cup E_i$, then E_i form a basis of the Picard group.

$\bar{K}_X = \sum a_i E_i$, where $\bar{K}_X = K_X + \sum E_i$

These a_i will be called $\bar{K}$-labels. They do not change under blow-ups, are invariants of divisiorial valuations val_{E_i}.

Building Rules:

1. When a point on one curve is blown up, the new curve's label is its parent's label plus 1.

2. When a point of intersection is blown up, the new curve's label is the sum of its parents' labels.

We will be blowing up points on both copies of P^2.

Properties:

1) The labels of neighbors are coprime.

2) If any of the parents has a postive label, then the label of the curve is positive.

3) The subgraph of vertices with negative labels is connected. It is separated from the “positive” vertices by the “zero” vertices. Moreover, the “zero” vertices are only connected to vertices with labels (-1) (exactly one such curve) or 1 (zero or more curves).

Adjunction Formula

Suppose $\phi : X \to Y$ is a resolution of a Keller map, $\tau : X \to W$, $\rho : W \to Y$ is its Stein factorization. Then

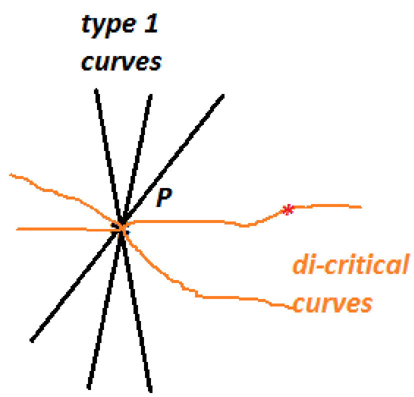
$$\bar{K}_W = \rho^*(\bar{K}_Y) + \bar{R},$$

where $\bar{R} = \sum r_i R_i$ is the di-critical ramification divisor: R_i are curves mapped to A^2, with ramification r_i.

Corollary. For every non-di-critical curve E, its label is the label of $\phi(E)$ times ramification index. In particular $\bar{K} - 0$ valuations go to $\bar{K} - 0$ valuations.

Question. What are the $\bar{K} - 0$ valuations (algebraically, non-inductively)?

Structure of W for $Y = P^2$



$$P = \tau(\pi_*^{-1}(\infty))$$

Theorem. $\bar{R}$ is ample.

Other properties: W is smooth outside of P and possible cyclic quotient singularities on the di-critical curves (up to one per curve). Type 1 curves are smooth outside P.

Determinant Labels

Definition. Suppose E_i is a curve at infinity. Then the determinant label of the divisorial valuation corresponding to E_i is the determinant of the Gram matrix of minus-intersection form on all curves at infinity of X, except E_i. That is,

$$d_{E_i} = \det\left(-E_j \cdot E_k\right)_{j,k \neq i}$$

Property. It is an invariant of a valuation (i.e. stable under blow-ups).

Building Rules:

1. When a point on one curve is blown up, the new curve's label is its parent's label minus 1.

2. When a point of intersection is blown up... **???**

Answer. Label the edges as well: d_{PQ} is the determinant of the graph with edge PQ removed.

Building Rules:

(Γ is a cycle-free weighted graph, $V(\Gamma)$ and $E(\Gamma)$ are sets of vertices and edges, $d = det(\Gamma)$).

1. $P \in V(\Gamma)$ is blown up, new vertex is R. Then

$$d_R = d_P + d, \quad d_{PR} = d_P + d$$

2. $PQ \in E(\Gamma)$ is blown up, new vertex is R. Then

$$d_R = 2d_{PQ} + d_P + d_Q - d,$$

$$d_{PR} = d_P + d_{PQ}, \quad d_{QR} = d_Q + d_{PQ}.$$

Importance of the Determinant Labels:

Any Keller map must send a valuation with positive det. label to a valuation with positive det. label.

Determinant labels of curves sent to the line at infinity are negative (moreover, "common determinant" is negative)

Any curve with negative $\bar{K}$ and determinant labels has a $\bar{K}-0$ "switchback" ancestor.

Theorem. For any fixed values of the $\bar{K}$ and determinant labels, there is N, such that any curve with these labels is obtained from P^2 by $\leq N$ blow-ups (boundedness upto automorphisms).

Theorem. If $\bar{K}$ label is -2 and determinant label is 1, it is a line at infinity for some P^2.

Local Structure of W, for any Y

Suppose $\rho : W \to Y$ is finite, $\rho(A^2) \subseteq A^2$, Y is a smooth toroidal compactification of A^2, i.e. $Y \setminus A^2$ has (simple) normal crossings.

Theorem. If E is a curve of type 1 on W, not intersecting any di-critical curves, then W and ρ are toroidal in the neighborhood of E. If images of all di-critical curves intersect the component at infinity transversally, then W and ρ are toroidal.

What is Next?

1) $\bar{K}-0$ curves hold a key. What are they? How are they mapped by Keller maps?

2) Database of toroidal compactifications (their graphs)?

3) Higher dimension?

4) **Collaborators?**

Thank You for Your Attention!