Convolution Structures and Arithmetic Cohomology

Alexander Borisov, University of Pittsburgh

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One of the most influential observations in mathematics is the analogy between algebraic geometry and number theory. It was discovered more than 100 years ago, yet we still don't really know how far it goes. The following "dictionary" is for the most part classical.

Basic Dictionary

Algebraic Geometry	**Number Theory**
Polynomials over a field, for example $\mathbb{C}[x]$	Integers
Rational functions, for example $\mathbb{C}(x)$	Rational numbers
Finite extensions of $\mathbb{C}(x)$, i.e. $\mathbb{C}(C)$	Finite extensions of $\mathbb{Q}$, $[K:\mathbb{Q}] < \infty$
Smooth complete algebraic curve C, a.k.a. compact Riemann surface	The set of all valuations, of K, including finite (non-Archimedean) and infinite (Archimedean)
Projective line $\mathbb{P}^1(\mathbb{C})$	$\overline{\mathrm{Spec}(\mathbb{Z})} = \{2, 3, 5, ...; \infty\}$
Affine line $\mathbb{A}^1(\mathbb{C})$	$\mathrm{Spec}(\mathbb{Z}) = \{2, 3, 5, ...\}$

Theory of Divisors

Algebraic Geometry	Number Theory
Divisor D on a curve C is a formal finite sum $\sum a_P[P]$, where P are points on C, $a_P \in \mathbb{Z}$	Arakelov divisor D on $\overline{\mathrm{Spec}\,K}$ is a formal finite sum $\sum a_v[v]$, where $a_v \in \mathbb{Z}$ for finite $[v]$, $a_v \in \mathbb{R}$ for infinite $[v]$
Degree of a divisor, $\deg(D) = \sum a_P$	Degree of a divisor, $\deg(D) =$ $= \sum_{v=v_{\mathcal{P}}} \ln\|\mathcal{O}_K/\mathcal{P}\|a_v + \sum_{v\ inf.} e_v a_v$
Divisor of a rational function (principal divisor) $(f) = \sum_{P\in C} ord_P(f)\cdot[P]$	Divisor of an algebraic number (principal Arakelov divisor) $(a) = \sum_{v=v_{\mathcal{P}}} ord_{\mathcal{P}}(a)[v_{\mathcal{P}}] - \sum_{v\,\mathrm{inf.}} \ln\|a\|_v[v]$
Rational Equivalence $D_1 \sim D_2 \Longleftrightarrow D_1 - D_2 = (f)$	Rational Equivalence $D_1 \sim D_2 \Longleftrightarrow D_1 - D_2 = (a)$
# of zeroes = # of poles Thm $\deg(f) = 0$	Product Formula (log version) $\deg(a) = 0$
Picard Group $\mathrm{P}ic(C)$: all divisors/principal divisors $\mathrm{P}ic^0(C) = \mathrm{J}ac(C)$: degree zero classes	Arakelov class group of K: Arakelov divisors/principal ones $\mathrm{P}ic^0(K)$: degree zero classes of Arakelov divisors
Theorem: $\mathrm{P}ic^0(C)$ is compact (abelian variety of dim g)	Dirichlet Unit Theorem and Finiteness of Ideal Class Group: $\mathrm{P}ic^0(K)$ is compact

Global Sections and Riemann-Roch Theorem

Algebraic Geometry	**Number Theory**
Space of Global Sections of $D = \sum_P a_P[P]$: $H^0(D) = \{f \mid (f) + D \geq 0\}$	Set of Global Sections of $D = \sum_v a_v[v]$: $H^0(D) = \{a \mid (a) + D \geq 0\}$
Dimension: $h^0(D) = \dim H^0(D)$ Theorem. $h^0(D) < \infty$	Dimension: $h^0(D) = \ln(\lvert H^0(D)\rvert)$ Theorem. $h^0(D) < \infty$
Example. $C = \mathbb{P}^1$, $D = [\pi] + 2[\infty]$ $H^0(D) = \{\frac{h(x)}{x-\pi} \mid \deg \frac{h(x)}{x-\pi} \leq 2\}$ $= \{\frac{c_3x^3+c_2x^2+c_1x+c_0}{x-\pi}\}$ $h^0(D) = 4 = \deg D + 1$	Example. $K = \mathbb{Q}$, $D = [3] + 2[\infty]$ $H^0(D) = \{\frac{n}{3} \mid n \in \mathbb{Z}, \ln \lvert\frac{n}{3}\rvert \leq 2\}$ $= \{-22, -21, ..., 21, 22\}$ $h^0(D) = \ln 45 \approx \deg D + \ln 2$
Canonical Class K_C: divisors of differential forms	Relative Canonical Class: δ_K^{-1}, where δ_K is the Different
Riemann-Roch Theorem $h^0(D) - h^0(K-D) = \deg D + 1 - g$ Corollary. For $\deg D > 2g-2$ $h^0(D) = \deg D + 1 - g$	Lattice Points Estimate $h^0(D) - h^0(K-D) \approx \deg D + 1 - g$ Corollary. For $\deg D \to +\infty$ $h^0(D) = \deg D + 1 - g + o(1)$
"Modern" Riemann-Roch $H^1(D)$; $h^1(D) = \dim H^1(D)$ $\chi(D) = h^0(D) - h^1(D)$ $\chi(D) = \deg D + \chi(0)$ Serre's Duality Theorem: $H^1(D)$ is dual to $H^0(K-D)$	Arithmetic Riemann-Roch ??? $\chi(D) = -\ln(covolume(I))$ $\chi(D) = \deg D + \chi(0)$ Too much to hope for: $\mathbb{Z}$ is too discrete!

Tate's Riemann Roch Formula

(van der Geer - Schoof version, 1999)

Idea: Instead of counting elements of $H^0(D)$ in the usual manner (1 if they are small, 0 is they are big), count them with weight $e^{-\pi Q(x)}$, where Q is a positive quadratic function on I. Here I is a fractional ideal generated by the restrictions from the finite places, and Q depends on the coefficients for the infinite places.

Example. $K = \mathbb{Q}$, $D = [3] + 2[\infty]$.

Then the fractional ideal $I = \frac{1}{3}\mathbb{Z}$. According to van der Geer and Schoof's convention, the function Q is given by $Q(x) = e^{-2a_\infty} \cdot x^2 = e^{-4}x^2$. For $x = \frac{n}{3}$ this gives

$$Q(x) = e^{-4}(\frac{n}{3})^2 = e^{-4-2\ln 3} \cdot n^2$$

So the dimension of D is

$$h^0(D) = \ln\left(\sum_{x\in I} e^{-\pi e^{-4}x^2}\right) = \ln\left(\sum_{n\in\mathbb{Z}} e^{-\pi e^{-4-2\ln 3}\cdot n^2}\right)$$

Theorem. (Tate's Riemann-Roch Formula).

$$h^0(D) - h^0(K-D) = \deg D - \frac{1}{2}\ln|\Delta|,$$

where Δ is the discriminant of K.

Proof: Poisson Summation Formula:

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{k\in\mathbb{Z}} \hat{f}(k)$$

For example, when $K = \mathbb{Q}$, every Arakelov divisor is rationally equivalent to $a[\infty]$. Then

$$h^0(a[\infty]) = \ln\left(\sum_{n\in\mathbb{Z}} e^{-\pi\cdot e^{-2a}\cdot n^2}\right)$$

The Riemann-Roch Theorem says

$$h^0(a[\infty]) - h^0(-a[\infty]) = a$$

Remark. As $a \rightarrow -\infty$, $h^0(-a[\infty])$ decreases very rapidly.

Remark. This equation is directly related to the functional equation for Riemann Zeta function.

Remark. This work resulted in interesting further development, in particular by Lagarias and Rains, 2003.

Question. We have correct $h^0(D)$. We can define $h^1(D) = h^0(K - D)$. What about $H^0(D)$, $H^1(D)$?

Idea: We have to consider objects that are more general than usual abelian groups.

G is a locally compact abelian group, H is a subgroup of G, i_H is its characteristic function.

(1) $i_H(0) = 1$

(2) i_H is even (i.e. $i_H(-x) = i_H(x)$ for all $x \in G$)

(3) i_H is positive-definite

(4) $(i_H)^2 = i_H$

The other way: for any such i_H, $i_H = 1$ is a subgroup of G.

To get H^0 to match van der Geer-Schoof's h^0, we must let go of (4).

Convolution of measures structures

G is a locally compact abelian group, two algebras:

(Functions, $\cdot$); (Measures, $*$)

Also, a pairing $(f, \mu) = \int_{x \in G} f(x) d\mu(x)$

A convolution of measures structure: a pair of algebras together with a pairing as above, not necessarily coming from a locally compact abelian group.

Note: analytical details vary; can be non-commutative

Pontryagin duality: Functions $\leftrightarrow$ Measures

Ghost-spaces

Formally: a triple (G, u, μ). Notation: G_u^μ

G : locally compact abelian group

u : positive, positive-definite function, $u(0) = 1$

μ : positive positive-definite probability measure on G

Intuitively: G_u^μ is the group G with "elements" that "partially exist" with probability u and/or are "imprecisely positioned", with error μ.

The convolution of measures structure:

$$\delta_x * \delta_y = \frac{u(x)u(y)}{u(x+y)} T_{x+y}\mu$$

$T_{x+y}\mu$ is the translation of μ by the element $(x + y)$

Note: $G_1^{\delta_0} = G$.

We omit the point measure δ_0 and the identity function 1 from the notation when possible:

$G_u \equiv G_u^{\delta_0}$, $G^\mu \equiv G_1^\mu$, $G \equiv G_1^{\delta_0}$.

The following picture represents the ghost-space $\mathbb{R}_{e^{-x^2}}$. One should think of it as being embedded into the usual real line.

The quotient $\mathbb{R}/\mathbb{R}_{e^{-x^2}}$ is the space $\mathbb{R}^\mu$, where μ is the probability measure proportional to $e^{-x^2}dx$.

Arithmetic Cohomology Theory

Only ghost-spaces of 1st kind, G_u and ghost-spaces of 2nd kind, G^μ were used.

Main Features

1) Some short exact sequences of ghost-spaces were introduced, with appropriate dimension function being additive.

2) $H^1(D)$: compact ghost-space of 2nd kind, definition resembles Čech cohomology. Then $h^1(D)$ is its (absolute) dimension.

3) The Tate's Riemann-Roch theorem was separated into the "modern" Riemann-Roch theorem

$$h^0(D) - h^1(D) = \deg D - \frac{1}{2}\deg K$$

and Serre's duality $\widehat{H^1(D)} = H^0(K - D)$.
(Thus $h^1(D) = h^0(K - D)$).

4) Serre's duality = Pontryagin duality of convolution structures.
Closely related to the Pontryagin duality theory of M. Rösler (1995)
Explicitly: $\widehat{G_u^\mu} = \hat{G}_{\check{\mu}}^{\hat{u}}$

An adèlic version: Ichiro Miyada (unpublished).

What's Next?

One basically gets the perfect theory in this one-dimensional case. Dimension two case is the famous Arakelov theory. Its basic setup is the following.

$X \to \mathrm{Spec}(\mathcal{O}_K)$, a complete curve over a ring of integers of some number field K

Each embedding of $\sigma : K \to \mathbb{C}$ gives a complex curve X_σ (curves at infinity).

An Arakelov divisor: formal finite linear combination of the following:
1) Vertical divisors: irreducible components of the fibers above prime ideals $\mathcal{P} \in \mathrm{Spec}(\mathcal{O}_K)$
2) Curves at infinity (with real coefficients)
3) Horizontal curves: L-rational points on X for a finite extension L of K.

The main innovation of Arakelov (early 1970's, inspired by previous work of Parshin): define the intersection of two horizontal divisors at infinity using the heat kernel on X_σ.

Further devlopments: Gerd Faltings, Shou-Wu Zhang, Henri Gillet, Christophe Soulé...
Arbitrary dimension. Grothendieck Riemann-Roch Theorem.
Euler characteristic only, suitably defined. No notion of $H^1(D)$ of $H^2(D)$.

Theories Relevant for Further Development

Parshin-Beilinson higher-dimensional adèles?

Sobolev spaces?

Brownian motion on manifolds?

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THANK YOU!