# Chapter 5

1. Which of the following statements about the real matrix shown below is FALSE?

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

   (a) $A$ is invertible
   (b) If $\mathbf{x} \in \mathbf{R}^5$ and $A\mathbf{x} = \mathbf{x}$, then $\mathbf{x} = \mathbf{0}$
   (c) The last row of $A^2$ is $\begin{bmatrix} 0 & 0 & 0 & 0 & 25 \end{bmatrix}$
   (d) $A$ can be transformed into the $5 \times 5$ identity matrix by a sequence of elementary row operations
   (e) $\det(A) = 120$

2. Let $A$ be a real $3 \times 3$ matrix. Which of the following conditions does NOT imply that $A$ is invertible?

   (a) $-A$ is invertible
   (b) There exists a positive integer $k$ such that $\det(A^k) \neq 0$
   (c) There exists a positive integer $k$ such that $(I - A)^k = 0$, where $I$ is the $3 \times 3$ identity matrix
   (d) The set of all vectors of the form $A\mathbf{v}$, where $\mathbf{v} \in \mathbf{R}^3$, is $\mathbf{R}^3$
   (e) There exist 3 linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbf{R}^3$ such that $A\mathbf{v}_i \neq \mathbf{0}$ for each $i$

3. Let $M$ be a $5 \times 5$ real matrix. Exactly four of the following five conditions on $M$ are equivalent to each other. Which of the five conditions is equivalent to NONE of the other four?

   (a) For any two distinct column vectors $u$ and $v$ of $M$, the set $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent
   (b) The homogeneous system $M\mathbf{x} = 0$ has only the trivial solution
   (c) The system of equations $M\mathbf{x} = \mathbf{b}$ has a unique solution for each real $5 \times 1$ column vector $\mathbf{b}$
   (d) The determinant of $M$ is nonzero
   (e) There exists a $5 \times 5$ real matrix $N$ such that $NM$ is the $5 \times 5$ identity matrix

4. Let $A, A'$ be an $n \times n$ matrix and let $A'$ be obtained from $A$ by elementary row transformations. Which of the following statements are correct?

   (a) $\det A = 0 \Leftrightarrow \det A' = 0$
   (b) $\det A = \det A'$
   (c) $\det A = \lambda \det A'$ for some $\lambda \in \mathbf{R}, \lambda \neq 0$

5. Which of the following assertions is correct? For an $n \times n$ matrix $A$ we have

   (a) $\det A = 0 \Rightarrow \text{rk } A = 0$

(b) det $A = 0 \Leftrightarrow$ rk $A \leq n - 1$

(c) det $A = 0 \Rightarrow$ rk $A = n$

6. Which of the following statements holds for all $n \times n$ matrices $A, B, C$ and all $\lambda \in \mathbf{R}$

   (a) det $(A + B) =$ det $A+$ det $B$

   (b) det $\lambda A = \lambda$ det $A$

   (c) det $((AB)C) =$ det $A$ det $B$ det $C$

7. det $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & -1 \\ 0 & 1 & 1 \end{bmatrix} =$

   (a) 2

   (b) 4

   (c) 6

8. det $\begin{bmatrix} \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda \end{bmatrix} =$

   (a) 0

   (b) $\lambda$

   (c) $\lambda^3$

9. det $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} =$

   (a) $\cos 2\theta$

   (b) 0

   (c) 1

10. Which of the following assertions is (or are) false?

    (a) det $A = 1 \Rightarrow A = E$

    (b) det $A = 1 \Rightarrow A$ is injective as a map $\mathbf{R}^n \to \mathbf{R}^n$

    (c) det $A = 1 \Rightarrow A$ is surjective as a map $\mathbf{R}^n \to \mathbf{R}^n$

11. Let $A$ be an $n \times n$ matrix and det $A = 0$. Then $Ax = b$ is

    (a) solvable only for $b = 0$

    (b) solvable for all $b$, possibly nonuniquely

    (c) solvable only for some $b$, and then never uniquely

12. What is the determinant of $\begin{bmatrix} 5 & 4 \\ 1 & 3 \end{bmatrix}$

(a) 4

(b) 11

(c) 15

(d) 19

13. What is the determinant of $\begin{bmatrix} 5 & 1 & 3 \\ 1 & 3 & 2 \\ 0 & -1 & 1 \end{bmatrix}$

(a) 0

(b) 15

(c) 24

(d) 26

14. What is the determinant of $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(a) 0

(b) 9

(c) 15

15. What is the determinant of $\begin{bmatrix} 5 & 2 & -1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

(a) 0

(b) 6

(c) 15

(d) 22

16. Which of the following matrices are not invertible?

(a) $\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} -2 & 3 \\ 2 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & -3 & 3 \\ -2 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

(e) More than one of the above

(f) All of the above have inverses

17. Suppose the determinant of a $2 \times 2$ matrix $A$ is equal to 3. What is the determinant of $A^{-1}$

    (a) $\frac{1}{3}$

    (b) 3

    (c) 9

    (d) Not enough information is given

18. Suppose the determinant of a $2 \times 2$ matrix $A$ is equal to 3. What is the determinant of $5A$?

    (a) 3

    (b) 9

    (c) 15

    (d) 75

    (e) Not enough information is given

19. If $A$ is a $2 \times 2$ matrix, then $\det(kA)$ is

    (a) $k \det(A)$

    (b) $2k \det(A)$

    (c) $k^2 \det(A)$

    (d) Not enough information given

20. Which of the following statements is true?

    (a) If a square matrix has two identical rows then its determinant is zero

    (b) If the determinant of a matrix is zero, then the matrix has two identical rows

    (c) Both are true

    (d) Neither is true

21. Suppose the determinant of matrix $A$ is zero. How many solutions does the system $Ax = b$ have?

    (a) 0

    (b) 1

    (c) Infinite

    (d) Not enough information is given

22. Suppose the determinant of matrix $A$ is zero. How many solutions does the system $Ax = 0$ have?

   (a) 0
   (b) 1
   (c) Infinite
   (d) Not enough information is given

23. Consider the matrix $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

   (a) The columns are linearly dependent
   (b) The matrix has determinant $-1$
   (c) The matrix is not invertible
   (d) None of the above

24. Assume that B is a $3 \times 3$ matrix with the property that $B^2 = B$. Which of the following statements about the matrix $B$ MUST be true:

   (a) B is invertible
   (b) $\det(B) = 0$
   (c) $\det(B^5) = \det(B)$
   (d) None of the above must be true

25. Let $C = \begin{bmatrix} 2 & -3 & 0 & 3 \\ 0 & -2 & 2 & -3 \\ 0 & 3 & -2 & -3 \\ 0 & -1 & 1 & -2 \end{bmatrix}$. What is $\det C$?

   (a) 2
   (b) $-3$
   (c) 0
   (d) 4
   (e) $-\frac{2}{5}$

26. Consider the matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$

   Which of the following statements is true?

   (a) The columns of $A$ are linearly dependent

(b) $\det(A) = 1$

(c) $A$ is not invertible

(d) None of the above statements apply

27. What is the determinant of the matrix $\begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 5 & 0 & 0 & 4 \end{bmatrix}$

(a) $-120$

(b) $-24$

(c) $0$

(d) $24$

(e) $120$

28. Let $A$, $B$, $C$ be square invertible matrices satisfying $AB = B^2C$. Assume that $\det B = 3$ and $\det C = 2$. Find a formula for $A$ and calculate the determinant of $A$

(a) $A = BC$, $\det A = 6$

(b) $A = B^3C$, $\det A = 11$

(c) $A = B^2CB^{-1}$, $\det A = 6$

(d) $A = B^2CB^{-1}$, $\det A = 5$

(e) $A = B^3C$, $\det A = 54$

(f) $A = BC$, $\det A = 5$

29. If $A$ and $B$ are square matrices of size $n \times n$, then which of the following statements is not true?

(a) $\det(AB) = \det(A) \det(B)$

(b) $\det(kA) = k^n \det(A)$

(c) $\det(A + B) = \det(A) + \det(B)$

(d) $\det(A^T) = 1/\det(A^{-1})$

30. What is the determinant of the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 100 & 1 & 0 & 0 \\ 100 & 200 & 1 & 0 \\ 100 & 200 & 300 & 1 \end{bmatrix}$?

(a) 100

(b) 200

(c) 1

(d) 0

31. Let $A$ be a $5 \times 5$ matrix with determinant 6. What is the determinant of $A^{-1}$ (the inverse of $A$)?

(a) 25/6
(b) 0
(c) 1
(d) 30
(e) 1/6
(f) 6
(g) Insufficient information to solve the question

32. Let $A$ be a $5 \times 5$ matrix with determinant 6, and let $B$ be a $5 \times 5$ matrix with determinant 4. What is the determinant of $AB$?

(a) 0
(b) 1/4
(c) 1/6
(d) 10
(e) 24
(f) 1
(g) Insufficient information to solve the question

33. Let $A$ be a $5 \times 5$ matrix with determinant 6, and let $B$ be a $5 \times 5$ matrix with determinant 4. What is the determinant of $A + B$?

(a) 24
(b) 1/6
(c) 0
(d) 1
(e) 1/4
(f) 10
(g) Insufficient information to solve the question

34. Let $A$ be a $4 \times 4$ matrix with determinant 3. What is the determinant of $-A$?

(a) 0
(b) $-1$
(c) 1
(d) $-3$
(e) 1/3
(f) 3
(g) Insufficient information to solve the question

35. Let $A$ be a $4 \times 4$ matrix with determinant 3. What is the determinant of $2A$?

(a) 3
(b) 48
(c) 24
(d) 12
(e) 6
(f) 196,608
(g) Insufficient information to solve the question

36. Let $A$ be a $4 \times 4$ matrix with determinant 3. What is the determinant of $A^T$ (the transpose of $A$)?

(a) 12
(b) 81
(c) $3^T$
(d) 1/3
(e) 27
(f) 3
(g) Insufficient information to solve the question

37. Let $A$ be a $4 \times 4$ matrix with determinant 3. Let $B$ be the matrix formed by swapping the second and third rows of $A$. What is $\det(B)$?

   (a) 2
   (b) 0
   (c) 3
   (d) $-3$
   (e) 6
   (f) 1/3
   (g) Insufficient information to solve the question

38. Let $A$ be a $4 \times 4$ matrix with determinant 3. Let $B$ be the matrix formed by multiplying the fourth row of $A$ by 2. What is $\det(B)$?

   (a) 3/2
   (b) 3
   (c) $-48$
   (d) 48
   (e) $-6$
   (f) 6
   (g) Insufficient information to solve the question

39. Let $A$ be a $4 \times 4$ matrix with determinant 3. Let $B$ be the matrix formed by dividing the fourth row of $A$ by 2. What is $\det(B)$?

   (a) $-3$
   (b) 3/16
   (c) 6
   (d) 3/4
   (e) 3
   (f) 3/2
   (g) Insufficient information to solve the question

40. Let $A$ be a $4 \times 4$ matrix with determinant 3. Let $B$ be the matrix formed by adding two copies of the third row to the first. What is $\det(B)$?

(a) 3
(b) 0
(c) 6
(d) $-6$
(e) 9
(f) $-3$
(g) Insufficient information to solve the question

41. Let $A$ be a $4 \times 4$ matrix with determinant 3. Let $B$ be the matrix formed by subtracting two copies of the third row from the first. What is the $\det(B)$?

(a) 0
(b) 3
(c) 6
(d) $-6$
(e) $-9$
(f) $-3$
(g) Insufficient information to solve the question

42. Let $A$ be a $4 \times 4$ matrix. The relationship between determinant and invertibility is

(a) $A$ is invertible if its determinant is zero, but not conversely.
(b) $A$ is invertible if and only if its determinant is zero.
(c) The determinant of $A$ is zero if $A$ is invertible, but not conversely.
(d) There is no connection between the two concepts.
(e) The determinant of $A$ is non-zero if $A$ is invertible, but not conversely.
(f) $A$ is invertible if its determinant is non-zero, but not conversely.
(g) $A$ is invertible if and only if its determinant is non-zero.

43. if $A$ is a $3 \times 5$ matrix, then the determinant of $A$ is

(a) A $3 \times 5$ matrix
(b) A $5 \times 3$ matrix
(c) A number (possibly non-zero)
(d) A subspace of $\mathbf{R}^3$
(e) A subspace of $\mathbf{R}^5$

(f) Zero

(g) Undefined

44. Of the numbers 2, 3, 5, which are eigenvalues of the matrix $\begin{bmatrix} 3 & 5 & 3 \\ 1 & 7 & 3 \\ 1 & 2 & 8 \end{bmatrix}$?

(a) None

(b) 2 and 3 only

(c) 2 and 5 only

(d) 3 and 5 only

(e) 2, 3, and 5

45. Let $V$ be a finite-dimensional real vector space and let $P$ be a linear transformation of $V$ such that $P^2 = P$. Which of the following must be true?

I. $P$ is invertible

II. $P$ is diagonalizable

III. $P$ is either the identity transformation or the zero transformation

(a) None

(b) I only

(c) II only

(d) III only

(e) II and III

46. Let $A$ be a $2 \times 2$ matrix for which there is a constant $k$ such that the sum of the entries in each row and each column is $k$. Which of the following must be an eigenvector of $A$?

I. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

II. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

III. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(a) I only

(b) II only

(c) III only

(d) I and II only

(e) I, II, and III

47. Let $A$ be a real $2 \times 2$ matrix. Which of the following statements must be true?

I. All of the entries of $A^2$ are nonnegative

II. The determinant of $A^2$ is nonnegative

III. If $A$ has two distinct eigenvalues, then $A^2$ has two distinct eigenvalues

(a) I only
(b) II only
(c) III only
(d) II and III only
(e) I, II, and III

48. Which of the following is the larger of the eigenvalues of the matrix $\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$?

(a) 4
(b) 5
(c) 6
(d) 10
(e) 12

49. In order to be able to discuss the "eigenvalues" of a linear map $f : V \to W$ at all, $f$ must be

(a) epimorphic (surjective)
(b) isomorphic (bijective)
(c) endomorphic ($V = W$)

50. The vector $v \neq 0$ is called an eigenvector for the eigenvalue $\lambda$ is $f(v) = \lambda v$. If instead $f(-v) = \lambda v$, then

(a) $-v$ is an eigenvector for the eigenvalue $\lambda$
(b) $v$ is an eigenvector for the eigenvalue $-\lambda$
(c) $-v$ is an eigenvector for the eigenvalue $-\lambda$

51. If $f : V \to V$ is an endomorphisim and $\lambda$ is an eigenvalue of $f$, then by the eigenspace $E_\lambda$ of $f$ corresponding to the eigenvalue $\lambda$, one understands

(a) the set of all eigenvectors for the eigenvalue $\lambda$
(b) the set consisting of all eigenvectors for the eigenvalue $\lambda$, together with the zero vector
(c) Ker $(\lambda Id)$

52. Which of the following three vectors is an eigenvector of $f = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} : \mathbf{R}^2 \to \mathbf{R}^2$?

(a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$

53. Let $f : V \to V$ be an endomorphism of a finite-dimensional vector space, and let $\lambda_1, \ldots, \lambda_r$ be the distinct eigenvalues of $f$. Then

    (a) $\dim \lambda_1 + \cdots + \dim E_{\lambda_r} = \lambda_1 + \cdots + \lambda_r$

    (b) $\dim \lambda_1 + \cdots + \dim E_{\lambda_r} \leq n$

    (c) $\dim \lambda_1 + \cdots + \dim E_{\lambda_r} > n$

54. Let $f : \xrightarrow{\cong} V$ be an automorphism of $V$ and $\lambda$ an eigenvalue of $f$. Then

    (a) $\lambda$ is also an eigenvalue of $f^{-1}$

    (b) $-\lambda$ is an eigenvalue of $f^{-1}$

    (c) $\frac{1}{\lambda}$ is an eigenvalue of $f^{-1}$

55. An endomorphism $f$ of an $n$-dimensional vector space is diagonalizable if and only if

    (a) $f$ has $n$ distinct eigenvalues

    (b) $f$ has only one eigenvalue whose geometric multiplicity equals $n$

    (c) $n$ equals the sum of the geometric multiplicities of the eigenvalues

56. The concepts of eigenvalue, eigenvector, eigenspace, geometric multiplicity, and diagonalizability have been defined for endomorphisms of (sometimes finite-dimensional) vector spaces $V$. Which further "general assumption" on $V$ have we tacitly made here?

    (a) $V$ is always a real vector space

    (b) $V$ is always a Euclidean vector space

    (c) no extra assumption; $V$ is just a vector space over $\mathbf{R}$

57. If $f, g : V \to V$ are endomorphims and there exists some $\varphi \in GL(V)$ with $f = \varphi g \varphi^{-1}$, then $f$ and $g$ have

    (a) the same eigenvalues

    (b) the same eigenvectors

    (c) the same eigenspaces

58. Compute the product $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

    (a) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$

(e) None of the above

59. Compute the product $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^2 \times \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(a) $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} 6 \\ 6 \end{bmatrix}$

(c) $\begin{bmatrix} 9 \\ 9 \end{bmatrix}$

(d) $\begin{bmatrix} 12\ 12 \end{bmatrix}$

(e) None of the above

(f) This matrix multiplication is impossible

60. Compute the product $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^4 \times \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(a) $\begin{bmatrix} 27 \\ 27 \end{bmatrix}$

(b) $\begin{bmatrix} 81 \\ 81 \end{bmatrix}$

(c) $\begin{bmatrix} 243 \\ 243 \end{bmatrix}$

(d) $\begin{bmatrix} 729 \\ 729 \end{bmatrix}$

(e) None of the above

61. For any integer $n$, what will this product be? $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^n \times \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(a) $\begin{bmatrix} 3n \\ 3n \end{bmatrix}$

(b) $3^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(c) $n^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(d) $3^n \begin{bmatrix} n \\ n \end{bmatrix}$

(e) $\begin{bmatrix} 3 \\ 3 \end{bmatrix}^n$

(f) More than one of the above

62. Suppose $A$ is an $n \times n$ matrix, $c$ is scalar, and $x$ is an $n \times 1$ vector. If $Ax = cx$, what is $A^2x$?

(a) $2cx$

(b) $c^2x$

(c) $cx$

(d) None of the above

63. Compute the product $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(a) $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 3 \\ -3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(d) $\begin{bmatrix} -3 \\ 3 \end{bmatrix}$

(e) None of the above

64. For any integer $n$, what will this product be? $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^n \times \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(a) $(-1)^n \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(b) $(-1)^n \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(c) $(-3)^n \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(d) $\begin{bmatrix} (-1)^n \\ (-1)^{n+1} \end{bmatrix}$

(e) None of the above

(f) More than one of the above

65. For any integer $n$, what will this product be? $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^n \times \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

(a) $3^n \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

(b) $2^n \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

(c) $6^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(d) $3^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(e) None of the above

(f) More than one of the above

66. For any integer $n$, what will this product be? $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^n \times \begin{bmatrix} -5 \\ 5 \end{bmatrix}$

(a) $3^n \begin{bmatrix} -5 \\ 5 \end{bmatrix}$

(b) $(-1)^n \begin{bmatrix} -5 \\ 5 \end{bmatrix}$

(c) $(-5)^n \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(d) $5 \begin{bmatrix} (-1)^n \\ (-1)^n \end{bmatrix}$

(e) None of the above

(f) More than one of the above

67. Compute the product $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

(a) $\begin{bmatrix} 3 \\ 15 \end{bmatrix}$

(b) $\begin{bmatrix} -1 \\ -5 \end{bmatrix}$

(c) $\begin{bmatrix} 11 \\ 7 \end{bmatrix}$

(d) $\begin{bmatrix} 7 \\ 11 \end{bmatrix}$

(e) None of the above

68. For any integer $n$, what will this product be? $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^n \times \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

(a) $11^n \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

(b) $7^n \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

(c) $\begin{bmatrix} 11^n \\ 7^n \end{bmatrix}$

(d) $\begin{bmatrix} 25 \\ 29 \end{bmatrix}$

(e) None of the above

(f) More than one of the above

69. Write the vector $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(a) $\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \frac{1}{2}\left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$

(c) $\begin{bmatrix} 1 \\ 5 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(d) None of the above

(e) More than one of the above

70. For any integer $n$, what will this product be? $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^n \times \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

(a) $-1 \times 3^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3 \times (-2)^n \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(b) $3 \times (-1)^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-2) \times 3^n \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(c) $3 \times 3^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-2) \times (-1)^n \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(d) $3 \times 3^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-1) \times (-2)^n \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(e) None of the above

(f) More than one of the above

71. Which of the following is an eigenvector of the matrix $\begin{bmatrix} 2 & -1 \\ -4 & -1 \end{bmatrix}$

(a) $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$

(e) None of the above

(f) More than one of the above

72. Which of the following is an eigenvector of the matrix $\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

(a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} -3 \\ -3 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ -2/3 \end{bmatrix}$

(e) None of the above

(f) More than one of the above

73. Suppose the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has an eigenvalue 1 with associated eigenvector $x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. What is $A^{50}x$?

(a) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(b) $\begin{bmatrix} a^{50} & b^{50} \\ c^{50} & d^{50} \end{bmatrix}$

(c) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(d) $\begin{bmatrix} 2^{50} \\ 3^{50} \end{bmatrix}$

(e) Way too hard to compute

74. Vector $x$ is an eigenvector of matrix $A$. If $x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $Ax = \begin{bmatrix} 4 \\ 12 \end{bmatrix}$, then what is the associated eigenvalue?

(a) 1
(b) 3
(c) 4
(d) Not enough information is given

75. Which of the following is an eigenvector of $A = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$? (You should be able to answer this by checking the vectors given, rather than by finding the eigenvectors of $A$ from scratch)

(a) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$
(b) $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$
(c) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
(d) None of the above

76. $\begin{bmatrix} 4/3 \\ 1 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$. What is the associated eigenvalue? (Think! Don't solve for all the eigenvalues and eigenvectors)

(a) 4/3
(b) 5
(c) -2

77. The matrix $A = \begin{bmatrix} -1 & 4 \\ 3 & 0 \end{bmatrix}$ has an eigenvalue 3 with associated eigenvector $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let $y = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Which of the following statements is true?

(a) $Ax = 3x$
(b) $Ay = 3y$
(c) For any scalars $c$ and $d$, $A(cx + dy) = 3(cx + dy)$
(d) All of the above are true
(e) Only (a) and (d) are true

78. The matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ has an eigenvalue 2 with associated eigenvectors $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Which of the following statements is true?

(a) $Ax = 2x$

(b) $Ay = 2y$

(c) For any scalars $c$ and $d$, $A(cx + dy) = 2(cx + dy)$

(d) For any nonzero scalars $c$ and $d$, $cx + dy$ is an eigenvector of $A$ corresponding to the eigenvalue 2.

(e) All of the above are true

(f) Only (a) and (d) are true

79. What does it mean if 0 is an eigenvalue of a matrix $A$?

    (a) The determinant of $A$ is zero

    (b) The columns of $A$ are linearly dependent

    (c) There are an infinite number of solutions to the system $Ax = 0$

    (d) All of the above

    (e) None of the above

80. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 6 \\ 0 & 4 & 2 \end{bmatrix}$ and note that all of the rows sum to six. Which of the following is true?

    (a) $w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of $A$

    (b) 6 is an eigenvalue of $A$

    (c) Both statements are true

    (d) Neither statement is true

81. If a vector $x$ is in the eigenspace of $A$ corresponding to $\lambda$, and $\lambda \neq 0$, then $x$ is

    (a) in the nullspace of the matrix $A$

    (b) in the nullspace of the matrix $A - \lambda I$

    (c) not the zero vector

    (d) More than one of the above correctly completes the sentence

82. Which of the following statements is correct?

    (a) The set of eigenvectors of a matrix $A$ forms the eigenspace of $A$

    (b) The set of eigenvectors of a matrix $A$ spans the eigenspace of $A$

    (c) Since any multiple of an eigenvector is also an eigenvector, the eigenspace always has infinite dimension

    (d) More than one of the above statements are correct

    (e) None of the above statements are correct

83. Which of the following statements is correct?

   (a) The set of eigenvectors of a matrix $A$ corresponding to a particular eigenvalue $\lambda_1$, together with the zero vector, forms the eigenspace of A corresponding to $\lambda_1$
   (b) An eigenspace corresponding to a non-repeated eigenvalue has dimension one
   (c) An eigenvalue of multiplicity two has a corresponding eigenspace of dimension two
   (d) All of the above statements are correct
   (e) Exactly two of the above statements are correct

84. What are the eigenvalues of $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

   (a) 2 and 3
   (b) 0 and 2
   (c) 0 and 3
   (d) 5 and 6

85. If $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ what is $D^5$

   (a) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$
   (b) $\begin{bmatrix} 10 & 0 \\ 0 & 15 \end{bmatrix}$
   (c) $\begin{bmatrix} 2^5 & 0 \\ 0 & 3^5 \end{bmatrix}$
   (d) Too hard to compute by hand

86. Why might we be interested in diagonalizing a matrix?

   (a) Because it is easy to find the eigenvalues of a diagonal matrix
   (b) Because it is easy to compute powers of a diagonal matrix
   (c) Both of these reasons

87. Which of the following statements are true?

   (a) An $n \times n$ matrix with $n$ linearly independent eigenvectors is diagonalizable
   (b) Any diagonalizable $n \times n$ matrix has $n$ linearly independent eigenvectors
   (c) Both are true
   (d) Neither is true

88. Which of the following statements are true?

(a) An $n \times n$ matrix with $n$ distinct eigenvalues is diagonalizable
(b) Any diagonalizable $n \times n$ matrix has $n$ distinct eigenvalues
(c) Both are true
(d) Neither is true

89. Which of the following statements are true?

(a) If $A$ is a diagonalizable matrix, then $A$ does not have any zero eigenvalues
(b) If $A$ does not have any zero eigenvalues, then $A$ is diagonalizable
(c) Both are true
(d) Neither is true

90. The number of linearly independent eigenvectors of the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

is given by:

(a) 0
(b) 1
(c) 2
(d) 3
(e) 4
(f) 5

91. $A = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 8 & -5 \\ -3 & 10 & -7 \end{bmatrix}$, $X = \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}$, $Y = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Which of the following statements is true?

(a) $Y$ is an eigenvector of $A$ with the eigenvalue 2
(b) $Y$ is an eigenvector of $A$ with the eigenvalue $-2$
(c) $Y$ is an eigenvector of $A$ with the eigenvalue 3
(d) $X$ is an eigenvector of $A$ with the eigenvalue 3
(e) $X$ is an eigenvector of $A$ with the eigenvalue $-2$
(f) $X$ is an eigenvector of $A$ with the eigenvalue 2

92. $A^2 - A = 0$, where $A$ is a $9 \times 9$ matrix. Then

(a) $A$ must be a zero matrix

(b) $A$ is an identity matrix

(c) Rank of $A$ is 1 or 0

(d) $A$ is diagonalizable

93. The sum of eigenvalues of $\begin{bmatrix} -1 & -2 & -1 \\ -2 & 3 & 2 \\ -1 & 2 & -3 \end{bmatrix}$ is

(a) $-3$

(b) $-1$

(c) 3

(d) 1

94. Let $A$ and $B$ are square matrices such that $AB = I$, then zero is an eigenvalue of

(a) $A$ but not of $B$

(b) $B$ but not of $A$

(c) Both $A$ and $B$

(d) Neither $A$ nor $B$

95. The matrix $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ is not diagonalizable over the reals. Why?

(a) Because $A$ does not have any real eigenvectors

(b) Because $A$ does not have any real eigenvalues

(c) Because $A$ does not have three distinct eigenvalues

(d) Because $A$ does not have three independent eigenvectors

(e) Because $A$ is lower triangular

(f) Because 'diagonalizable' is too difficult to say very quickly

96. What is/are the eigenvector(s) of the matrix $\begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(a) $(0, 0, a)$

(b) $(a, 0, 0)$

(c) $(0, 0, 1)$

(d) $(0, a, 0)$

(e) More than one of the above (specify which)

97. Let $A$ be an $n \times n$ matrix. Which of the following criteria will ensure that $A$ is diagonalizable over the reals?

(a) The characteristic polynomial of $A$ splits over the reals

(b) The characteristic polynomial of $A$ has no repeated roots

(c) $A$ can be row reduced to a diagonal matrix

(d) The rows of $A$ are linearly independent.

(e) The determinant of $A$ is non-zero

(f) $A$ has $n$ distinct real eigenvalues

(g) $A$ commutes with all diagonal matrices

98. Let $A$ be an $n \times n$ matrix with real entries. Which of the following statements is *not* necessarily true?

(a) $\det(A)$ is non-zero

(b) $A$ is similar to the identity matrix

(c) $A$ can be written as the product of elementary row matrices

(d) One can row reduce $A$ to the identity matrix

(e) Zero is not an eigenvalue of $A$

(f) The null space of $A$ contains only the zero vector

(g) The linear transformation $x \to Ax$, from $\mathbf{R}^n$ to $\mathbf{R}^n$, is both one-to-one and onto

99. Let $A$ be an invertible $5 \times 5$ matrix. Which of the following statements is *false*?

(a) All the eigenvalues of $A$ are non-zero

(b) The kernel of $A$ is 0

(c) The columns of $A$ form a basis for $\mathbf{R}^5$

(d) The determinant of $A$ is non-zero

(e) The image of $A$ is $\mathbf{R}^5$

(f) The rows of $A$ form a basis for $\mathbf{R}^5$

(g) There are five distinct eigenvalues

100. Let $A$ be an diagonalizable $5 \times 5$ matrix. Which one of the following statements is *false*?

(a) The total algebraic multiplicities of all the eigenvalues must equal 5

(b) The algebraic multiplicity of each eigenvalue must equal the geometric multiplicity of each eigenvalue

(c) The total geometric multiplicities of all the eigenvalues must equal 5

(d) All the eigenvalues must be distinct (occur with algebraic and geometric multiplicity 1)

(e) The determinant of A must equal the product of all the eigenvalues (counted with multiplicity)

(f) There must exist a basis of $\mathbf{R}^5$ consisting entirely of eigenvectors of $A$

(g) The rank of A is equal to the number of non-zero eigenvalues (counted with multiplicity)

101. Let $A$ be an $n \times n$ matrix. Which of the following criteria will ensure that $A$ is diagonalizable over the reals?

  (a) $A$ can be row reduced to a diagonal matrix
  (b) The characteristic polynomial of $A$ has no repeated costs
  (c) The rows of $A$ are linearly independent
  (d) $A$ has $n$ distinct real eigenvalues
  (e) The determinant of $A$ is non-zero
  (f) The characteristic polynomial of $A$ splits over the reals

102. Let $A$ be an invertible $5 \times 5$ matrix. Which of the following statements is *false*?

  (a) The image of $A$ is $\mathbf{R}^5$
  (b) The rows of $A$ form a basis for $\mathbf{R}^5$
  (c) The determinant of $A$ is non-zero. The kernel of $A$ is 0.
  (d) All the eigenvalues of $A$ are non-zero
  (e) There are five distinct eigenvalues
  (f) The columns of $A$ form a basis for $\mathbf{R}^5$

103. Let $A$ be an $n \times n$ invertible matrix with real entries. Which of the following statements is *NOT* necessarily true?

  (a) One can row reduce $A$ to the identity matrix
  (b) $A$ is similar to the identity matrix
  (c) $A$ can be written as the product of elementary matrices
  (d) Zero is not an eigenvalue of $A$
  (e) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$, from $\mathbf{R}^n$ to $\mathbf{R}^n$, is both one-to-one and onto
  (f) The null space of $A$ contains only the zero vector
  (g) $\det(A)$ is non-zero