Chapter 6

Directions: For questions 1 - 20, all vectors are in $\mathbf{R}^n$. Mark each statement True or False. Justify each answer.

1. (**True** | **False**) $\mathbf{v} \cdot \mathbf{v} = ||\,\mathbf{v}\,||^2$
2. (**True** | **False**) For any scalar c, $\mathbf{u} \cdot (c\,\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
3. (**True** | **False**) If the distance from $\mathbf{u}$ to $\mathbf{v}$ equals the distance from $\mathbf{u}$ to $-\,\mathbf{v}$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal
4. (**True** | **False**) For a square matrix A, vectors in Col A are orthogonal to vectors in Nul A
5. (**True** | **False**) If vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span a subspace W and if $\mathbf{x}$ is orthogonal to each $\mathbf{v}_j$ for $j = 1, \ldots, p$, then $\mathbf{x}$ is in $W^\perp$
6. (**True** | **False**) $\mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} = 0$
7. (**True** | **False**) For any scalar c, $||c\,\mathbf{v}\,|| = c||\,\mathbf{v}\,||$
8. (**True** | **False**) If $\mathbf{x}$ is orthogonal to every vector in a subspace W, then $\mathbf{x}$ is in $W^\perp$
9. (**True** | **False**) If $||\,\mathbf{u}\,||^2 + ||\,\mathbf{v}\,||^2 = ||\,\mathbf{u} + \mathbf{v}\,||^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal
10. (**True** | **False**) For an $m \times n$ matrix A, vectors in the null space of A are orthogonal to vectors in the row space of A
11. (**True** | **False**) Not every linearly independent set in R^n is an orthogonal set
12. (**True** | **False**) If $\mathbf{y}$ is a linear combination of nonzero vectors from an orthogonal set, then the weights in the linear combination can be computed without row operations on a matrix
13. (**True** | **False**) If the vectors in an orthogonal set of nonzero vectors are normalized, then some of the new vectors may not be orthogonal
14. (**True** | **False**) A matrix with orthonormal columns is an orthogonal matrix
15. (**True** | **False**) If L is a line through $\mathbf{0}$ and if $\hat{\mathbf{y}}$ is the orthogonal projection of $\mathbf{y}$ onto L, then $||\hat{\mathbf{y}}||$ gives the distance from $\mathbf{y}$ to L
16. (**True** | **False**) Not every orthogonal set in $\mathbf{R}^n$ is linearly independent
17. (**True** | **False**) If a set $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ has the property that $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$, then S is an orthonormal set
18. (**True** | **False**) If the columns of an $m \times n$ matrix A are orthonormal, then the linear mapping $\mathbf{x} \mapsto A\,\mathbf{x}$ preserves lengths
19. (**True** | **False**) The orthogonal projection of $\mathbf{y}$ onto $\mathbf{v}$ is the same as the orthogonal projection of $\mathbf{y}$ onto $c\,\mathbf{v}$ whenever $c \neq 0$

20. (**True** | **False**) An orthogonal matrix is invertible

Directions: For questions 21 - 35, all vectors and subspaces are in $\mathbf{R}^n$. Mark each statement True or False. Justify each answer.

21. (**True** | **False**) If $\mathbf{z}$ is orthogonal to $\mathbf{u}_1$ and to $\mathbf{u}_2$ and if $W = \text{Span}\,\{\mathbf{u}_1, \mathbf{u}_2\}$, then $\mathbf{z}$ must be in $W^\perp$

22. (**True** | **False**) For each $\mathbf{y}$ and each subspace W , the vector $\mathbf{y} - \text{proj}_W\,\mathbf{y}$ is orthogonal to W

23. (**True** | **False**) The orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto a subspace W can sometimes depend on the orthogonal basis for W used to compute $\hat{\mathbf{y}}$

24. (**True** | **False**) If $\mathbf{y}$ is in a subspace W, then the orthogonal projection of $\mathbf{y}$ onto W is $\mathbf{y}$ itself

25. (**True** | **False**) If the columns of an $n \times p$ matrix U are orthonormal, then $UU^T\mathbf{y}$ is the orthogonal projection of $\mathbf{y}$ onto the column space of U

26. (**True** | **False**) If W is a subspace of $\mathbf{R}^n$ and if $\mathbf{v}$ is in both W and $W^\perp$ then $\mathbf{v}$ must be the zero vector

27. (**True** | **False**) If $\mathbf{y} = \mathbf{z}_1 + \mathbf{z}_2$, where $\mathbf{z}_1$ is in a subspace W and $\mathbf{z}_2$ is in $W^\perp$, then $\mathbf{z}_1$ must be the orthogonal projection of $\mathbf{y}$ onto W

28. (**True** | **False**) The best approximation to $\mathbf{y}$ by elements of a subspace W is given by the vector $\mathbf{y} - \text{proj}_W\,\mathbf{y}$

29. (**True** | **False**) If an $n \times p$ matrix U has orthonormal columns, then $UU^T\mathbf{x} = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbf{R}^n$

30. (**True** | **False**) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for W, then multiplying $\mathbf{v}_3$ by a scalar c gives a new orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, c\,\mathbf{v}_3\}$

31. (**True** | **False**) The Gram–Schmidt process produces from a linearly independent set $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ an orthogonal set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ with the property that for each k, the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ span the same subspace as that spanned by $\mathbf{x}_1, \ldots, \mathbf{x}_k$

32. (**True** | **False**) If $A = QR$, where Q has orthonormal columns, then $R = Q^TA$

33. (**True** | **False**) If $W = \text{Span}\,\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ with $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ linearly independent, and if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set in W, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for W

34. (**True** | **False**) If $\mathbf{x}$ is not in a subspace W, then $\mathbf{x} - \text{proj}_W\,\mathbf{x}$ is not zero

35. (**True** | **False**) In a QR factorization, say $A = QR$ (when A has linearly independent columns), the columns of Q form an orthonormal basis for the column space of A

Directions: For questions 36 - 46, A is an $m \times n$ matrix and $\mathbf{b}$ is in $\mathbf{R}^m$. Mark each statement True or False. Justify each answer.

36. (**True** | **False**) The general least-squares problem is to find an $\mathbf{x}$ that makes $A\,\mathbf{x}$ as close as possible to $\mathbf{b}$

37. (**True** | **False**) A least-squares solution of $A\mathbf{x} = \mathbf{b}$ is a vector $\hat{\mathbf{x}}$ that satisfies $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$, where $\hat{\mathbf{b}}$ is the orthogonal projection of $\mathbf{b}$ onto Col A

38. (**True** | **False**) A least-squares solution of $A\mathbf{x} = \mathbf{b}$ s a vector $\hat{\mathbf{x}}$ such that $||\mathbf{b} - A\mathbf{x}|| \leq ||\mathbf{b} - A\hat{\mathbf{x}}||$ for all $\mathbf{x}$ in $\mathbf{R}^n$

39. (**True** | **False**) Any solution of $A^T A\mathbf{x} = A^T\mathbf{b}$ is a least-squares solution of $A\mathbf{x} = \mathbf{b}$

40. (**True** | **False**) If the columns of A are linearly independent, then the equation $A\mathbf{x} = \mathbf{b}$ has exactly one least-squares solution

41. (**True** | **False**) If $\mathbf{b}$ is in the column space of A, then every solution of $A\mathbf{x} = \mathbf{b}$ is a least-squares solution

42. (**True** | **False**) The least-squares solution of $A\mathbf{x} = \mathbf{b}$ is the point in the column space of A closest to $\mathbf{b}$

43. (**True** | **False**) A least-squares solution of $A\mathbf{x} = \mathbf{b}$ is a list of weights that, when applied to the columns of A, produces the orthogonal projection of $\mathbf{b}$ onto Col A

44. (**True** | **False**) If $\hat{\mathbf{x}}$ is a least-squares solution of $A\mathbf{x} = \mathbf{b}$, then $\hat{\mathbf{x}} = (A^T A)^{-1} A^T\mathbf{b}$

45. (**True** | **False**) The normal equations always provide a reliable method for computing least-squares solutions

46. (**True** | **False**) If A has a QR factorization, say $A = QR$, then the best way to find the least-squares solution of $A\mathbf{x} = \mathbf{b}$ is to compute $\hat{\mathbf{x}} = R^{-1}Q^T\mathbf{b}$

Directions: Questions 47 - 65 refer to vectors in $\mathbf{R}^n$ (or $\mathbf{R}^m$) with the standard inner product. Mark each statement True or False. Justify each answer.

47. (**True** | **False**) The length of every vector is a positive number.

48. (**True** | **False**) A vector $\mathbf{v}$ and its negative, $-\mathbf{v}$, have equal lengths.

49. (**True** | **False**) The distance between $\mathbf{u}$ and $\mathbf{v}$ is $||\mathbf{u} - \mathbf{v}||$.

50. (**True** | **False**) If r is any scalar, then $||r\mathbf{v}|| = r||\mathbf{v}||$.

51. (**True** | **False**) If two vectors are orthogonal, they are linearly independent.

52. (**True** | **False**) If $\mathbf{x}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{x}$ must be orthogonal to $\mathbf{u} - \mathbf{v}$.

53. (**True** | **False**) If $||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

54. (**True** | **False**) If $||\mathbf{u} - \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

55. (**True** | **False**) The orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$ is a scalar multiple of $\mathbf{y}$.

56. (**True** | **False**) If a vector $\mathbf{y}$ coincides with its orthogonal projection onto a subspace W, then $\mathbf{y}$ is in W.

57. (**True** | **False**) The set of all vectors in $\mathbf{R}^n$ orthogonal to one fixed vector is a subspace of $\mathbf{R}^n$.

58. (**True** | **False**) If W is a subspace of $\mathbf{R}^n$, then W and $W^\perp$ have no vectors in common.

59. (**True** | **False**) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set and if c_1, c_2, and c_3 are scalars, then $\{c_1\,\mathbf{v}_1, c_2\,\mathbf{v}_2, c_3\,\mathbf{v}_3\}$ is an orthogonal set.

60. (**True** | **False**) If a matrix U has orthonormal columns, then $UU^T = I$.

61. (**True** | **False**) A square matrix with orthogonal columns is an orthogonal matrix.

62. (**True** | **False**) If a square matrix has orthonormal columns, then it also has orthonormal rows.

63. (**True** | **False**) If W is a subspace, then $||\text{proj}_W\,\mathbf{v}\,||^2 + ||\,\mathbf{v} - \text{proj}_W\,\mathbf{v}\,||^2 = ||\,\mathbf{v}\,||^2$.

64. (**True** | **False**) A least-squares solution of $A\,\mathbf{x} = \mathbf{b}$ is the vector $A\hat{\mathbf{x}}$ in Col A closest to $\mathbf{b}$, so that $||\,\mathbf{b} - A\hat{\mathbf{x}}|| \leq ||\,\mathbf{b} - A\,\mathbf{x}\,||$ for all $\mathbf{x}$.

65. (**True** | **False**) The normal equations for a least-squares solution of $A\,\mathbf{x} = \mathbf{b}$ are given by $\hat{\mathbf{x}} = (A^TA)^{-1}A^T\,\mathbf{b}$

For questions 66 - 74, mark each statement True or False. Justify your answer.

66. (**True** | **False**) If two vectors are linearly independent, they must be orthogonal

67. (**True** | **False**) Any orthogonal set of nonzero vectors that spans a vector space must be a basis for that space

68. (**True** | **False**) Let M be any matrix. The columns of M are orthonormal if and only if M^TM is an identity matrix

69. (**True** | **False**) Let Q be a square matrix with orthonormal columns, then $Q^{-1} = Q^T$

70. (**True** | **False**) Any set of nonzero orthogonal vectors must also be linearly independent

71. (**True** | **False**) The only orthonormal basis for R^2 is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

72. (**True** | **False**) If $W = \text{Span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, then $S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix} \right\}$ is an orthogonal basis for W

73. (**True** | **False**) For a nonzero $\mathbf{v}$ in $\mathbf{R}^n$, the matrix $\mathbf{v}\,\mathbf{v}^T$ is called a projection matrix

74. (**True** | **False**) An orthogonal matrix is orthogonally diagonalizable