Math 304 Class 18 March 4

Midterm 1: March 6 7pm.

Rooms: Unusual start time or other accommodation go to LH 008. (or UTC, as agreed with instructor.) Sections 4, 5, 6, 8 : LH008.

Other students, including most in Sections 1, 2, 3, and 7 : SI 149.

Bring your I.D. and a #2 pencil.

No books, notes, or calculators.

Today: Solving $MX = V$ using a LU decomposition.

Example: Given the LU decomposition.

$$\begin{pmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 6 & 0 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 6 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Use this to solve: $MX = V$ where

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad V = \begin{pmatrix} 3 \\ 19 \\ 0 \end{pmatrix}$$

Idea: solve 2 triangular systems, because solving triangular systems is easy.

$MX = V$ is $LUX = V$

define $W = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = UX$. The system.

$LW = V$ is triangular:

$$\begin{aligned} 3u &= 3 \\ u + 6v &= 19 \\ 2u + 3v + w &= 0 \end{aligned}$$

Solve by "forward-substitution": $u = 1$.

So $1 + 6v = 19$ $\quad 6v = 18 \quad v = 3$.

So $2 \cdot 1 + 3 \cdot 3 + w = 0 \quad w = -11$.

How do we get from $W_0 = \begin{pmatrix} 1 \\ 3 \\ -11 \end{pmatrix}$ to a solution?

Now $UX = W_0$ is also a triangular system: $\begin{pmatrix} 2 & 6 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -11 \end{pmatrix}$

$z = -11$ $\quad y = 3$ $\quad 2x + 6y + z = 1$

So $2x + 6 \cdot 3 + (-11) = 1$

$2x + 7 = 1$

$2x = -6$ $\quad$ $x = -3$

Solution: $X = \begin{pmatrix} -3 \\ 3 \\ -11 \end{pmatrix}$.

Note: This method is not obvious. Memorize the method. See box p. 161.

The previous example gave us the LU decomposition. What if we are asked to find the LU decomposition?

A: We have to do a row reduction, AND keep track of the operations.

Recall that row operations are implemented using left mult. by elementary matrices. So if E_1, E_2, E_3 are lower-triangular elementary matrices such that

$$E_3 E_2 E_1 M = U \quad \text{(upper-triangular)}.$$

So $(E_3 E_2 E_1)^{-1} (E_3 E_2 E_1) M = (E_3 E_2 E_1)^{-1} U$

or $$M = (E_3 E_2 E_1)^{-1} U = \underbrace{E_1^{-1} E_2^{-1} E_3^{-1}} U.$$

Lower-triangular

Let's try a VERY simple example.

$$M = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$$

E_1: add $-3 \times$ row 1 to row 2.

$$E_1 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$$

⚠ Possible test question: what elementary matrix corresponds to ~~that~~ a given ERO?

$$E_1 M = \begin{pmatrix} 1 & 2 \\ 0 & -5 \end{pmatrix} \leftarrow \text{upper-}\Delta\text{. "U".}$$

$$M = E_1^{-1} U = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -5 \end{pmatrix}.$$

Check: $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = M$ ✓

Now more complex example:

Find an LU decomposition for M

$$M = \begin{pmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{pmatrix}$$

Add $-\frac{1}{3} \times R1$ to $R2$ $\quad E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$$E_1 M = \begin{pmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 4 & 15 & 3 \end{pmatrix}$$

Add $-\frac{2}{3} \times R1$ to $R3$ $\quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{2}{3} & 0 & 1 \end{pmatrix}$

$$E_2 E_1 M = \begin{pmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 3 & 1 \end{pmatrix}$$

Add $-\frac{1}{2} \times R2$ to $R3$. $\quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}$

$$E_3 E_2 E_1 M = \begin{pmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix} = U.$$

Now $M = (E_3 E_2 E_1)^{-1} U = E_1^{-1} E_2^{-1} E_3^{-1} U.$

⚠ Note: $(AB)^{-1} = B^{-1} A^{-1}$.

Now we find $L = E_1^{-1} E_2^{-1} E_3^{-1}$

⚠ Note: easy to find inverses of elementary matrices. $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ "Add $1/3 \times R1$ to $R2$"

$$E_1^{-1} E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2/3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & 0 & 1 \end{pmatrix}$$

And finally:

$$E_1^{-1}E_2^{-1}E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{pmatrix}$$

$E_1^{-1}E_2^{-1}$ $\qquad$ E_3^{-1}

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & 1/2 & 1 \end{pmatrix} = L.$$

So our LU decomp is:

$$\begin{pmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & 1/2 & 1 \end{pmatrix}\begin{pmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

M $\qquad$ L $\qquad$ U

Note that the LU decomposition is not unique. We saw earlier a different LU decomp for this same M.

Example Midterm 1. (#1).

Solution set (text): $\left\{ \begin{pmatrix} 11/5 \\ 3/5 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -3/5 \\ 1/5 \\ 1 \end{pmatrix} : \mu \in \mathbb{R} \right\}$

Alt. solution: $\left\{ \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} : t \in \mathbb{R} \right\}$.

Note that the names of the variables and the numbers don't matter! (p. 46).

Only the SET matters, and these two sets are the same:

They are both lines and the direction is not affected by the magnitude of the direction vector:

$$5 \begin{pmatrix} -3/5 \\ 1/5 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix}$$

so the directions are the same.

Two lines in the same direction that have a point in common are the same line. So just check that

$\begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$ is on the other line:

$$\begin{pmatrix} 11/5 \\ 3/5 \\ 0 \end{pmatrix} + (-3)\begin{pmatrix} -3/5 \\ 1/5 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} \checkmark$$

2.5.1 Geometry of Solution Sets: Hyperplanes.

What to know?

#1 Suggestion: homework... there are many problems of the form "find the equation of a plane such that..." or "what do these equations mean?"

#2: Understand the situation in $\mathbb{R}^2$ and $\mathbb{R}^3$.

See p. 64-65.

Idea: an equation $ax+by+cz=d$ defines a plane. More generally

$a_1x_1+a_2x_2+\cdots+a_nx_n=b$ reduces dimension by 1, from n to $n-1$. It defines a "hyperplane": a copy of $\mathbb{R}^{n-1}$ in $\mathbb{R}^n$.

$ax+by+cz=d$ defines a copy of $\mathbb{R}^2$ in $\mathbb{R}^3$.

What I mean by #2: understand all the

ways that planes can intersect in $\mathbb{R}^3$.

There are 5 possibilities, listed ~~in~~ on p. 65.

How about the ways lines can intersect in $\mathbb{R}^2$? There are 4 possibilities.

Can you list them?

In $\mathbb{R}^n$ there are $(n+2)$ possibilities for the intersection of some hyperplanes. (p. 65).