Math 304 Class 40 May 3

Quiz solutions next page.

<u>Remarks on</u> # 3: The rank of an $m \times n$ matrix A, which we regard as a linear transformation $A: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is the number of pivot columns of A. Computing this number requires a row reduction: when A is put in RREF we can count the pivot columns.

The matrices A, B, C, D of the quiz are already in RREF, so there is no work to do.

The nullity is the dimension of the nullspace.

Observe that e.g. $B\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

So the nullspace of B is $\left\{ \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \middle| z \in \mathbb{R} \right\}$

which has dimension 1.

Notice that this is the same as the number of free variables, or non-pivot columns.

Math 304, Section 5 — Quiz 20 – May 3

Name: Solutions

1. Complete the following definition of the *rank* of a linear transformation $L\colon V \to W$: The rank of L is the dimension of ...

the image of L, which is a subspace of W.

2. Complete the following definition of the *nullity* of a linear transformation $L\colon V \to W$: The nullity of L is the dimension of ...

the kernel of L, which is a subspace of V

3. Find the rank and nullity of the following matrices, regarded as linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A) = 3$ $\quad$ $\text{rank}(B) = 2$ $\quad$ $\text{rank}(C) = 1$ $\quad$ $\text{rank}(D) = 0$

$\text{null}(A) = 0$ $\quad$ $\text{null}(B) = 1$ $\quad$ $\text{null}(C) = 2$ $\quad$ $\text{null}(D) = 3$.

We computed $B\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

When is $B\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$?

this requires $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ i.e. $x = 0$ and $y = 0$.

When $x = y = 0$, we have $B\begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

This works for any z, so $\text{null}(B) = \left\{ \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \middle| z \in \mathbb{R} \right\}$

Similarly, $\text{null}(C) = \left\{ \begin{pmatrix} 0 \\ y \\ z \end{pmatrix} \middle| y, z \in \mathbb{R} \right\}$

has dimension 2. This is the same as the number of free variables, or non-pivot columns of C.

For any of these matrices

rank + nullity $= 3 =$ dimension of domain.

This is true because the number of pivot columns + number of non-pivot columns = total columns = 3.

The real work in the rank-nullity theorem is seeing that the 2 definitions of rank:

1. dimension of the image.

2. number of pivot columns of the matrix.

are in fact the same.

Notice that if we had a linear transformation $L: V \to W$ between abstract vector spaces, we could choose bases B and C for V and W and ask about the number of pivot columns of ${}_C L_B$ (matrix of L wrt B, C).

<u>Example:</u> $V = W = P_2$ = polyn. of deg ≤ 2.

$L: P_2 \to P_2$ is $\frac{d}{dx}$.

What is the image of L?

```
Script started on Mon 01 Feb 2016 01:29:35 PM EST
lagrange:~/ComputerAlgebra/Maple$ maple

|\^/|     Maple 18 (IBM INTEL LINUX)
._|\|   |/|_. Copyright (c) Maplesoft, a division of Waterloo Maple Inc. 2014
 \  MAPLE  /  All rights reserved. Maple is a trademark of
 <____ ____>  Waterloo Maple Inc.
      |       Type ? for help.
> solve({v + 2*w + 4*x + 7*y + 3*z = 20,
>       -2*v -2*w -6*x -10*y + 3*z = 2,
>       -1*v + 0*w -2*x -3*y + 3*z = 10,
>       -3*v -2*w -8*x -13 *y + 3*z = 0,
>       -3*v + 0*w - 6*x -9*y + 3*z = 6},
>    [v,w,x,y,z]);


[[v = 2 - 2 x - 3 y, w = 3 - x - 2 y, x = x, y = y, z = 4]]



> solve({v + 2*w + 4*x + 6*y + 3*z = 20,
>       -2*v -2*w -7*x -10*y + 3*z = 2,
>       -1*v + 0*w -2*x -3*y + 3*z = 10,
>       -3*v -2*w -8*x +13 *y + 3*z = 3,
>       -3*v + 0*w - 6*x -9*y + 3*z = 6},
>    [v,w,x,y,z]);

                       47       147       -3
                 [[v = --, w = ---, x = --, y = 3/25, z = 4]]
                       25       50        25

>
> quit;
memory used=0.7MB, alloc=6.3MB, time=0.06
lagrange:~/ComputerAlgebra/Maple$ exit
exit

Script done on Mon 01 Feb 2016 01:30:30 PM EST
```

```
Script started on 2019-05-03 11:04:40-04:00
:~/ComputerAlgebra/R$ R

R version 3.5.1 (2018-07-02) -- "Feather Spray"
Copyright (C) 2018 The R Foundation for Statistical Computing
Platform: x86_64-redhat-linux-gnu (64-bit)

R is free software and comes with ABSOLUTELY NO WARRANTY.
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  Natural language support but running in an English locale

R is a collaborative project with many contributors.
Type 'contributors()' for more information and
'citation()' on how to cite R or R packages in publications.

Type 'demo()' for some demos, 'help()' for on-line help, or
'help.start()' for an HTML browser interface to help.
Type 'q()' to quit R.

> A = matrix(sample(-11:11,9),3,3)
> A
     [,1] [,2] [,3]
[1,]    8    9    1
[2,]   -4  -11   -2
[3,]   -1    6   -6
> eigen(A)
eigen() decomposition
$values
[1] -7.440034+3.33792i -7.440034-3.33792i  5.880069+0.00000i

$vectors
                      [,1]                   [,2]              [,3]
[1,]  0.1031661-0.2296437i  0.1031661+0.2296437i  0.9609755+0i
[2,] -0.1857887+0.4322297i -0.1857887-0.4322297i -0.2058183+0i
[3,]  0.8457426+0.0000000i  0.8457426+0.0000000i -0.1848377+0i

> quit();
Save workspace image? [y/n/c]: n
:~/ComputerAlgebra/R$ exit
exit

Script done on 2019-05-03 11:05:32-04:00
```

A: the space P_1 of polynomials of degree ≤ 1. ④

Why? Notice that the derivative reduces the ~~deg~~ degree. So the image is contained in P_1.

Notice that $\frac{d}{dx}(x) = 1$ $\quad \frac{d}{dx}\left(\frac{1}{2}x^2\right) = x$.

So, by linearity we can get any degree 1 polynomial.

Thus $\text{rank}(L) = \dim(P_1) = 2$.

What ~~a~~ is the kernel of L?

A: The constant polynomials P_0.

$$\frac{d}{dx}\left(ax^2+bx+c\right) = 2ax+b$$

this is the zero polyn. iff $a=b=0$.

Thus $\text{nullity}(L) = \dim(P_0) = 1$.

Notice $\quad \text{rank}(L) + \text{null}(L) = \dim(P_2)$.

$$2 \quad + \quad 1 \quad = \quad 3.$$

Now let's look at the matrix of L.

$B = (1, x, x^2)$

Find ${}_B L_B$ (matrix of L wrt B).

The first col. is the coeff of $L(1)$ wrt B.

- 2^{nd} ———————— $L(x)$ ————

- 3^{rd} ———————— $L(x^2)$ ————

$$L(1) = 0 = 0 \cdot 1 + 0x + 0x^2$$

$$L(x) = 1 = 1 \cdot 1 + 0x + 0x^2$$

$$L(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0x^2$$

$${}_B L_B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

When we put this in RREF there will be 2 pivot columns (rank 2!)

and 1 non-pivot column. (nullity 1!)

Why is this always the same?

Let A be an $m \times n$ matrix.

$x \longmapsto Ax$ gives a linear transformation $\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m$

What is the image of A?

<u>Ans</u>: all possible linear combinations of columns of A — this is the same as:

the span of the columns of A.

which is called the "column space of A"

sometimes "$\mathrm{col}(A)$".

Why? <u>Ans</u>: rules of matrix mult.

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + cy \\ bx + dy \end{pmatrix} = x\begin{pmatrix} a \\ b \end{pmatrix} + y\begin{pmatrix} c \\ d \end{pmatrix}$$

(This generalizes to other sizes.)

What is the kernel of A?

<u>Ans</u>: all vectors $x \in \mathbb{R}^n$ so that $Ax = 0$.

Notice that the kernel of A is not changed by EROs. If E is the matrix effecting an ERO, then $Ax = 0 \Rightarrow EAx = 0$

and $EAx = 0 \Rightarrow E^{-1}EAx = 0$

$\Rightarrow Ax = 0$

The kernel of A is the set of all linear relations among the columns of A.

So the set of linear relations among columns of A is not changed by EROs.

So linear independence of columns is not changed by EROs.

So the sets of columns which form a basis for the column space are not changed by EROs.

So if a set of columns forms a basis for the column space of the RREF, then that set of columns forms a basis for the column space of the original matrix.

This is true even though these column spaces will in general be different.

What set of columns forms a basis for the column space of a matrix in RREF?

e.g. $\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ Ans: the set of pivot cols.

So the corresponding set of columns forms a basis for the column space of the original matrix.

So rank = # of pivot cols

= rank = dimension of image.

We observed already that ~~col(A)~~ the null-space of A is NOT changed by EROs. Thus the nullity of A is not changed by putting A in RREF. - in this form it is obvious that the nullity is # free variables.

Upshot: We did the work to see that the two concepts of rank

1. dimension of image
2. # pivot cols

are the same.

Nullity is both

1. dimension of null-space
2. # of non-pivot columns.

So the proof of rank-nullity now is

pivot cols + # non-pivot cols = total cols

= dimension of domain.