Math 330 • Number Systems | Name ____________________

**Test 1 SOLUTIONS March 2023** | Instructor: Dikran Karagueuzian

1. Consider the implication $P \Rightarrow Q$ and answer the following questions.

   (a) (1 point) What is the converse of $P \Rightarrow Q$?

   > **Solution:** $Q \Rightarrow P$

   (b) (1 point) What is the contrapositive of $P \Rightarrow Q$?

   > **Solution:** $\neg Q \Rightarrow \neg P$

   (c) (4 points) Write down a truth table for $P \Rightarrow Q$.

   **Solution:**

   | $P$ | $Q$ | $(P \Rightarrow Q)$ |
   |---|---|---|
   | $T$ | $T$ | T |
   | $T$ | $F$ | F |
   | $F$ | $T$ | T |
   | $F$ | $F$ | T |

   (d) (6 points) The original statement $P \Rightarrow Q$, its converse, and its contrapositive, give rise to 3 pairs of statements: (original, converse), (original, contrapositive), and (converse, contrapositive). Which, if any, of these 3 pairs of statements are logically equivalent statements?

   > **Solution:** The original and contrapositive are logically equivalent statements. The converse is *not* logically equivalent to the original nor is it logically equivalent to the contrapositive.

   (e) (3 points) Is the statement $\neg P \land Q$ logically equivalent to any of the statements considered in the previous parts? If so, which ones? (No explanation is necessary to get credit for this question.)

   > **Solution:** No. Consider the following assignment of truth values: $P = Q = T$. Then we see that $\neg P \land Q$ is false, because $\neg P$ is false. But the original, converse, and contrapositive are all true.

2. Find a "useful denial" of the following statements. Reminder: a useful denial is a statement logically equivalent to the denial of the original statement. Be sure to handle

quantifiers appropriately. Write your useful denial in mathematically precise English, not in logical notation. *Hint: you need not think about whether these statements are true.*

(a) (5 points) For all functions $f$ such that $f$ has a maximum at $x_0$ and $f$ is differentiable at $x_0$, $f'(x_0) = 0$.

**Solution:** There exists a function $f$ such that $f$ has a maximum at $x_0$ and $f$ is differentiable at $x_0$, for which $f'(x_0) \neq 0$.

(b) (5 points) For all integers $j$ and $k$, if $n = jk$, then $j = 1$ or $k = 1$.

**Solution:** There exist integers $j$ and $k$ and an integer $n$ such that $n = jk$, but $j \neq 1$ and $k \neq 1$.

3. A sequence $(b_n)_{n=0}^{\infty}$ satisfies $b_0 = 1, b_1 = -2$, and $b_{n+1} = -3b_n - 2b_{n-1}$.

(a) (1 point) Find $b_2$.

**Solution:**

$$b_2 = -3b_1 - 2b_0 = -3 \cdot (-2) - 2 \cdot 1 = 6 - 2 = 4$$

(b) (1 point) Find $b_3$.

**Solution:**

$$b_3 = -3b_2 - 2b_1 = -3 \cdot (4) - 2 \cdot (-2) = -12 + 4 = -8$$

(c) (1 point) Find $b_4$.

**Solution:**

$$b_4 = -3b_3 - 2b_2 = -3 \cdot (-8) - 2 \cdot (4) = 24 - 8 = 16$$

(d) (8 points) Conjecture a "closed-form" formula for $b_n$.

**Solution:**

$$b_n = (-1)^n 2^n$$

(e) (2 points) Find $b_5$, and check that your formula gives the correct value for $b_5$.

**Solution:**

$$b_5 = -3b_4 - 2b_3 = -3 \cdot (16) - 2 \cdot (-8) = -48 + 16 = -32 = (-1)^5 2^5$$

(f) (2 points) Find $b_6$, and check that your formula gives the correct value for $b_6$.

**Solution:**

$$b_6 = -3b_5 - 2b_4 = -3 \cdot (-32) - 2 \cdot (16) = 96 - 32 = 64 = (-1)^6 2^6$$

4. A sequence $(a_n)_{n=1}^{\infty}$ satisfies $a_1 = 1, a_2 = 5$, and $a_{n+1} = a_n + 2a_{n-1}$. In this problem you will prove by induction that $a_n = 2^n + (-1)^n$.

(a) (2 points) Identify a statement $P(n)$ to be proved.

**Solution:** Let $P(n)$ be the statement that $a_n = 2^n + (-1)^n$.

(b) (2 points) Complete the base step of the induction.

**Solution:** For the base step we will show that $a_1 = 2^1 + (-1)^1 = 2 - 1 = 1$ and that $a_2 = 2^2 + (-1)^2 = 4 + 1 = 5$. This is true by the definition of the sequence $(a_n)_{n=1}^{\infty}$.

(c) (8 points) Complete the induction step.

**Solution:** For the induction step, we will show for all $n \geq 2$ that $P(n-1)$ and $P(n)$ together imply $P(n+1)$. Since $P(n-1)$ is true, we have $a_{n-1} = 2^{n-1} + (-1)^{n-1}$. Since $P(n)$ is true, we have $a_n = 2^n + (-1)^n$. By definition of the sequence $(a_n)_{n=1}^{\infty}$, $a_{n+1} = a_n + 2a_{n-1}$. Putting these facts together, we have

$$\begin{aligned} a_{n+1} &= 2^n + (-1)^n + 2 \cdot (2^{n-1} + (-1)^{n-1}) \\ &= 2^n + (-1)^{n-1} \cdot (-1) + 2 \cdot 2^{n-1} + 2 \cdot (-1)^{n-1} \quad = 2^n + 2^n + (-1)^{n-1}((-1) + 2) \\ &= 2 \cdot 2^n + (-1)^{n-1} \cdot 1 \\ &= 2^{n+1} + (-1)^{n-1} \cdot (-1)^2 \qquad = 2^{n+1} \cdot (-1)^{n+1} \end{aligned}$$

This is exactly $P(n+1)$, so we have completed the proof of the induction step.

(d) (1 point) In your proof, did you use strong induction?

> **Solution:** Yes, we used $P(n-1)$ and not just $P(n)$ to show $P(n+1)$.

5. Consider carefully each of the following propositions and attempted proofs. Indicate what is wrong with each attempted proof, if anything. (Some proofs may be correct.) *Hint: you are not being asked whether the propositions themselves are true. You are being asked to find the errors, if any, in the proofs.*

(a) (5 points) **Proposition:** Let $m \in \mathbb{Z}$. If $m \neq 0$ then $m^2 \in \mathbb{N}$.

*Attempted Proof.* Assume, seeking a contradiction, that $m = 0$. Then we have $m^2 = 0$. But $0 \notin \mathbb{N}$ by definition of natural numbers. This is a contradiction, proving the desired result. $\square$

> **Solution:** In a proof by contradiction, one assumes the negation of the statement to be proved. This attempted proof assumes instead the negation of the hypothesis $m \neq 0$. Also, the statement that is claimed to be a contradiction is not in fact a contradiction on its own, it is only contradicts the conclusion $m^2 \in \mathbb{N}$ of the statement to be proved.

(b) (5 points) **Proposition:** Let $m \in \mathbb{Z}$. If $m \neq 0$ then $m^2 \in \mathbb{N}$.

*Attempted Proof.* We may assume that $m \in \mathbb{N}$ or that $-m \in \mathbb{N}$, using the basic properties of integers. In the first case, take $m = 1$, then $m^2 = 1 \in \mathbb{N}$. In the second case, take $m = -1$, and again $m^2 = 1 \in \mathbb{N}$. $\square$

> **Solution:** The statement "If $m \neq 0$" has an implicit "for all" quantifier. This proof incorrectly takes this quantifier to be a "there exists" instead. The special cases $m = 1, -1$ are only sufficient to prove a "there exists" version of the statement and not the "for all" version.

(c) (8 points) **Proposition:** For every nonnegative integer $n$, $5n = 0$.

*Attempted Proof.* Let $P(n)$ be the statement $5n = 0$. We will prove this by strong induction.

For the base step, we have to prove that $5 \cdot 0 = 0$. This is true by basic properties of integers.

For the induction step, we show that if $P(i)$ is true for all $i$ such that $0 \leq i \leq m$, then $P(m+1)$ is true also. Let $i$ and $j$ be integers such that $i + j = m + 1$ and $0 \leq i \leq m$ and $0 \leq j \leq m$. By the induction hypothesis and the definition of $i$ and $j$, $P(i)$ and $P(j)$ are true. By definition of $P(i)$ and $P(j)$, we have $5i = 0$ and $5j = 0$. Adding these equations, $5i + 5j = 5(i + j) = 5(m + 1) = 0$. This proves $P(m + 1)$.

This completes the proof of the induction step (PSMI 2). Thus $P(n)$ is true for all $n \geq 0$.

**Solution:** The proof of the induction step $P(m) \Rightarrow P(m+1)$ fails when $m = 0$, because there are no integers $i$ and $j$ satisfying the given conditions. More specifically, when $m = 0$, the proof claims that there are integers $i, j$ satisfying $i + j = 0 + 1$ and $0 \leq i \leq 0$ and $0 \leq j \leq 0$. These conditions are $i = 0$, $j = 0$, and $i + j = 1$, which is impossible. Thus the proof that $P(0) \Rightarrow P(1)$ is incorrect, and the proof of the induction step is not correct.

6. Are the following statements true or false? If the statement is true, you should provide a short proof. If the statement is false, you should provide a counterexample.

   (a) (6 points) $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})[xy > 0]$

   **Solution:** False. For a counterexample, let $x \in \mathbb{R}$ be given, and take $y = -x$. Then $xy = -x^2 \leq 0$ cannot satisfy $xy > 0$.

   (b) (6 points) $(\forall x \in \mathbb{Q})(\forall y \in \mathbb{Q})(\exists z \in \mathbb{Q})[x < z < y]$

   **Solution:** False. If $y \leq x$, there is no $z$ such that $x < z < y$.

   (c) (6 points) $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(\exists z \in \mathbb{Z})[(y > x) \wedge (y + z = 500)]$.

   **Solution:** True. Let $y = x + 1$ and $z = 500 - y$. Then $y > x$ and $y + z = 500$.

7. Say whether or not the following statements are true, and give a proof or counterexample, as appropriate.

   (a) (7 points) The product of two *distinct* irrational numbers is irrational.

   **Solution:** False. We have $(2 + \sqrt{2})(2 - \sqrt{2}) = 2^2 - 2 = 2$, a rational number. But $(2 + \sqrt{2})$ and $(2 - \sqrt{2})$ are distinct irrational numbers, because the sum of a rational number and an irrational number is irrational.

   For completeness we prove that the sum of a rational number $\frac{a}{b}$ and an irrational number $x$ is irrational. Seeking a contradiction, assume that $\frac{a}{b} + x = \frac{c}{d}$ is rational. Then $x = \frac{c}{d} - \frac{a}{b} = \frac{bc-ad}{bd}$ is rational, a contradiction.

   (b) (7 points) The product of a nonzero rational number and an irrational number is irrational.

**Solution:** True. Assume, seeking a contradiction, that the product of a rational number $\frac{a}{b}$ and an irrational number $x$ is rational. Then $\frac{a}{b} \cdot x = \frac{c}{d}$ is rational. Therefore $x = \frac{c}{d}\frac{b}{a} = \frac{bc}{ad}$ is rational, a contradiction.

8. In this problem, you may use the fact (to be proved in Chapter 6) that for all integers $n$, if $n$ is not divisible by 3, then there exists an integer $k$ such that $n = 3k + 1$ or $n = 3k + 2$. You may also use without proof the fact that if $n$ is of the form $3k + 1$ or $3k + 2$ for some integer $k$, then $n$ is not divisible by 3.

   (a) (6 points) Show that if $n$ is not divisible by 3, then there is an integer $l$ such that $n^2 = 3l + 1$.

   **Solution:** If $n$ is not divisible by 3, then there exists a $k$ such that $n = 3k + 1$ or $n = 3k + 2$. In the first case, $n^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1 = 3l + 1$ where $l = 3k^2 + 2k$ is an integer since $k$ is.

   In the second case, $n^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1 = 3l + 1$ where $l = 3k^2 + 4k + 1$ is an integer since $k$ is.

   So such an integer $l$ always exists, proving the desired result.

   (b) (6 points) Show that if $3|(i^2 + j^2)$, then $3|i$ and $3|j$.

   **Solution:** We will prove the contrapositive: if $3 \nmid i$ or $3 \nmid j$, then $3 \nmid (i^2 + j^2)$. Since $3 \nmid i$ or $3 \nmid j$, one of $3 \nmid i$, $3 \nmid j$ must be true.

   Suppose first that $3 \nmid i$. Then we have $i^2 = 3l+1$ for some integer $l$ by part (a). Now either $3 \mid j$ or $3 \nmid j$.

   If $3 \mid j$, then we have $j = 3m$ for some integer $m$, and $i^2 + j^2 = 3l + 1 + 9m^2 = 3(l + 3m^2) + 1$ is not divisible by 3, using the facts on divisibility by 3 given in the statement of the problem. If $3 \nmid j$, by part (a) we have $j^2 = 3n+1$ for some integer $n$ and $i^2 + j^2 = 3l + 1 + 3n + 1 = 3(l + n) + 2$ is not divisible by 3, again using the facts given in the statement of the problem.

   Thus we have shown that if $3 \nmid i$, then $i^2 + j^2$ is not divisible by 3.

   Now we must show that if $3 \nmid j$, then $i^2 + j^2$ is not divisible by 3.

   If $3 \nmid j$, we have $j^2 = 3l + 1$ for some integer $l$. We consider two cases: either $3 \mid i$ or $3 \nmid i$.

   If $3 \mid i$, then we have $i = 3m$ for some integer $m$, and $i^2 + j^2 = 9m^2 + 3l + 1 = 3(3m^2 + 3l) + 1$ is not divisible by 3, using the facts on divisibility by 3 given in the statement of the problem. If $3 \nmid i$, by part (a) we have $i^2 = 3n + 1$ for some integer $n$ and $i^2 + j^2 = 3n + 1 + 3l + 1 = 3(n + l) + 2$ is not divisible by 3, again using the facts given in the statement of the problem.

Thus we have shown that if $3 \nmid j$, then $i^2 + j^2$ is not divisible by 3.

This completes the proof of the contrapositive, which is logically equivalent to the original statement. Thus we have completed the proof.