Math 330 • Number Systems

Test 1, March 1, 2023

Name ____________________________

Instructor: Dikran Karagueuzian

Question	Points	Score
1	15	
2	10	
3	15	
4	13	
5	18	
6	18	
7	14	
8	12	
Total:	115	

1. Consider the implication $P \Rightarrow Q$ and answer the following questions.
   (a) (1 point) What is the converse of $P \Rightarrow Q$?

   (b) (1 point) What is the contrapositive of $P \Rightarrow Q$?

   (c) (4 points) Write down a truth table for $P \Rightarrow Q$.

   (d) (6 points) The original statement $P \Rightarrow Q$, its converse, and its contrapositive, give rise to 3 pairs of statements: (original, converse), (original, contrapositive), and (converse, contrapositive). Which, if any, of these 3 pairs of statements are logically equivalent statements?

   (e) (3 points) Is the statement $\neg P \wedge Q$ logically equivalent to any of the statements considered in the previous parts? If so, which ones? (No explanation is necessary to get credit for this question.)

2. Find a "useful denial" of the following statements. Reminder: a useful denial is a statement logically equivalent to the denial of the original statement. Be sure to handle quantifiers appropriately. Write your useful denial in mathematically precise English, not in logical notation. *Hint: you need not think about whether these statements are true.*

   (a) (5 points) For all functions f such that f has a maximum at x_0 and f is differentiable at x_0, $f'(x_0) = 0$.

   (b) (5 points) For all integers j and k, if $n = jk$, then $j = 1$ or $k = 1$.

3. A sequence $(b_n)_{n=0}^{\infty}$ satisfies $b_0 = 1, b_1 = -2$, and $b_{n+1} = -3b_n - 2b_{n-1}$.

(a) (1 point) Find b_2.

(b) (1 point) Find b_3.

(c) (1 point) Find b_4.

(d) (8 points) Conjecture a "closed-form" formula for b_n.

(e) (2 points) Find b_5, and check that your formula gives the correct value for b_5.

(f) (2 points) Find b_6, and check that your formula gives the correct value for b_6.

4. A sequence $(a_n)_{n=1}^{\infty}$ satisfies $a_1 = 1, a_2 = 5$, and $a_{n+1} = a_n + 2a_{n-1}$. In this problem you will prove by induction that $a_n = 2^n + (-1)^n$.

   (a) (2 points) Identify a statement $P(n)$ to be proved.

   (b) (2 points) Complete the base step of the induction.

   (c) (8 points) Complete the induction step.

   (d) (1 point) In your proof, did you use strong induction?

5. Consider carefully each of the following propositions and attempted proofs. Indicate what is wrong with each attempted proof, if anything. (Some proofs may be correct.) *Hint: you are not being asked whether the propositions themselves are true. You are being asked to find the errors, if any, in the proofs.*

   (a) (5 points) **Proposition:** Let $m \in \mathbb{Z}$. If $m \neq 0$ then $m^2 \in \mathbb{N}$.

   Attempted Proof. Assume, seeking a contradiction, that $m = 0$. Then we have $m^2 = 0$. But $0 \notin \mathbb{N}$ by definition of natural numbers. This is a contradiction, proving the desired result. □

   (b) (5 points) **Proposition:** Let $m \in \mathbb{Z}$. If $m \neq 0$ then $m^2 \in \mathbb{N}$.

   Attempted Proof. We may assume that $m \in \mathbb{N}$ or that $-m \in \mathbb{N}$, using the basic properties of integers. In the first case, take $m = 1$, then $m^2 = 1 \in \mathbb{N}$. In the second case, take $m = -1$, and again $m^2 = 1 \in \mathbb{N}$. □

(c) (8 points) **Proposition:** For every nonnegative integer n, $5n = 0$.

Attempted Proof. Let $P(n)$ be the statement $5n = 0$. We will prove this by strong induction.

For the base step, we have to prove that $5 \cdot 0 = 0$. This is true by basic properties of integers.

For the induction step, we show that if $P(i)$ is true for all i such that $0 \leq i \leq m$, then $P(m+1)$ is true also. Let i and j be integers such that $i + j = m + 1$ and $0 \leq i \leq m$ and $0 \leq j \leq m$. By the induction hypothesis and the definition of i and j, $P(i)$ and $P(j)$ are true. By definition of $P(i)$ and $P(j)$, we have $5i = 0$ and $5j = 0$. Adding these equations, $5i + 5j = 5(i + j) = 5(m + 1) = 0$. This proves $P(m + 1)$.

This completes the proof of the induction step (PSMI 2). Thus $P(n)$ is true for all $n \geq 0$.

6. Are the following statements true or false? If the statement is true, you should provide a short proof. If the statement is false, you should provide a counterexample.

   (a) (6 points) $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})[xy > 0]$

   (b) (6 points) $(\forall x \in \mathbb{Q})(\forall y \in \mathbb{Q})(\exists z \in \mathbb{Q})[x < z < y]$

   (c) (6 points) $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(\exists z \in \mathbb{Z})[(y > x) \wedge (y + z = 500)]$.

7. Say whether or not the following statements are true, and give a proof or counterexample, as appropriate.

   (a) (7 points) The product of two *distinct* irrational numbers is irrational.

   (b) (7 points) The product of a nonzero rational number and an irrational number is irrational.

8. In this problem, you may use the fact (to be proved in Chapter 6) that for all integers n, if n is not divisible by 3, then there exists an integer k such that $n = 3k + 1$ or $n = 3k + 2$. You may also use without proof the fact that if n is of the form $3k + 1$ or $3k + 2$ for some integer k, then n is not divisible by 3.

   (a) (6 points) Show that if n is not divisible by 3, then there is an integer l such that $n^2 = 3l + 1$.

   (b) (6 points) Show that if $3|(i^2 + j^2)$, then $3|i$ and $3|j$.