Math 330 • Number Systems

Test 2, April 24, 2023

Name ____________________________

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Question	Points	Score
1	13	
2	10	
3	13	
4	28	
5	16	
6	18	
7	31	
Total:	129	

1. A sequence $(a_n)_{n=1}^{\infty}$ satisfies $a_1 = a_2 = a_3 = 1$ and $a_{n+1} = a_n + a_{n-1} + a_{n-2}$ for $n \geq 3$. In this problem you will prove by induction that $a_n < 2^n$ for all $n \geq 1$.

   (a) (2 points) Identify a statement $P(n)$ to be proved.

   (b) (2 points) Complete the base step of the induction.

   (c) (8 points) Complete the induction step.

   (d) (1 point) In your proof, did you use strong induction?

2. Consider carefully each of the following propositions and attempted proofs. Indicate what is wrong with each attempted proof, if anything. (Some proofs may be correct.) *Hint: you are not being asked whether the propositions themselves are true. You are being asked to find the errors, if any, in the proofs.*

   (a) (5 points) **Proposition:** For any positive integer n, $\sum_{i=1}^{n} i = \frac{1}{2}(n + \frac{1}{2})^2$.

   Attempted Proof. We will prove this by induction. Let $P(n)$ be the statement that $\sum_{i=1}^{n} i = \frac{1}{2}(n + \frac{1}{2})^2$.

   For the base step, a calculation shows that $P(n)$ is true for $n = 1$.

   For the induction step, we show that if $P(n)$ is true then $P(n+1)$ is true. This is again a calculation: assuming $P(n)$ we have $\sum_{i=1}^{n} i = \frac{1}{2}(n + \frac{1}{2})^2$. Now

$$\begin{aligned} \sum_{i=1}^{n+1} i &= \frac{1}{2}(n + \frac{1}{2})^2 + (n+1) \\ &= \frac{1}{2}(n^2 + n + \frac{1}{4} + 2n + 2) \\ &= \frac{1}{2}((n^2 + 2n + 1) + (n+1) + \frac{1}{4}) \\ &= \frac{1}{2}((n+1) + \frac{1}{2})^2 \end{aligned}$$

   This final equality gives us $P(n+1)$, so we have proved $P(n) \Rightarrow P(n+1)$ and the proof is complete.

(b) (5 points) **Proposition:** Any positive integer m factors uniquely into a product of primes.

Attempted Proof. We prove this by strong induction on m.

For the base step, $m = 1$, m has no prime factors and the only possible factorization is $m = 1$.

For the induction step, if m is a prime, then m is its own factorization, and no other factors are possible by definition of prime.

If m is not prime, then m must have a factor l which is less than m, so $m = kl$ for some $k < m$, $l < m$. Now apply the induction hypothesis to k and l: the factorization of m is (factorization of l) $\cdot$ (factorization of k), so again m factors into primes. Since there is only one factorization of l and only one factorization of k, there is only one factorization of m.

Thus m factors uniquely into primes. This completes the induction step and so completes the proof.

3. Answer the following questions about sets A and B.

   (a) (3 points) Is it possible that both $A \in B$ and $A \subseteq B$ are true? (Only a yes or no answer is required.)

   (b) (10 points) Give an example of such sets, or a proof that no such sets A and B exist to support your answer to the first part.

4. Let X, Y, Z be sets and $f\colon X \to Y$, $g\colon Y \to Z$ be functions.

(a) (4 points) Suppose $g \circ f$ is one-to-one. Must f be one-to-one? Must g be one-to-one? (Only yes/no answers are necessary.)

(b) (10 points) Give proofs or examples to justify your answers to the previous part.

(c) (4 points) Suppose $g \circ f$ is onto. Must f be onto? Must g be onto? (Only yes/no answers are necessary.)

(d) (10 points) Give proofs or examples to justify your answers to the previous part.

5. Let X, Y be sets and let $f\colon X \to Y$ be a function.

(a) (8 points) Suppose that C and D are subsets of Y. Prove that $f^{-1}[C \cap D] = f^{-1}[C] \cap f^{-1}[D]$.

(b) (8 points) Suppose A and B are subsets of X. Either prove that $f[A \cap B] = f[A] \cap f[B]$, or give a counterexample to show that this equality need not be true.

6. Let $\mathbb{Z}^+$ denote the positive integers and consider the function $f\colon \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by $f(m,n) = 2^{m-1}(2n-1)$. In what follows you may use the Fundamental Theorem of Arithmetic (FTA), but you must quote it explicitly if you do.

   (a) (9 points) Is f one-to-one? Prove that your answer is correct.

   (b) (9 points) Is f onto? Prove that your answer is correct.

7. Answer the following questions about arithmetic mod 59.

(a) (7 points) Show that there is a positive integer x such that $x \cdot 47 \equiv 1 \mod 59$.

(b) (12 points) Find a number m in $\{0, 1, 2, \ldots, 58\}$ which is a representative of $[x]_{59}$, i.e. a number such that $m \equiv x \mod 59$.

(c) (12 points) Find, with proof, the product $3 \cdot 4 \cdot 5 \cdots 56 \cdot 57 \mod 59$. Give an answer in $\{0, 1, 2, \ldots, 58\}$.