Math 447 - Probability

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BINGHAMTON
UNIVERSITY
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Chapter 1

Introduction

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Statistical techniques play an important role in achieving the objective of each of these practical situations.

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- Graphically, e.g. using a histogram to plot relative frequencies of, say, GPAs of students in the class, or
- Numerically, e.g. finding the average annual rainfall in California over the past 50 years and the deviation from this average quantity in a particular year.

We may also be interested in the *likelihood* of a certain event, e.g. drawing the Royalty (King and Queen) of *different* suits from a standard deck of cards.

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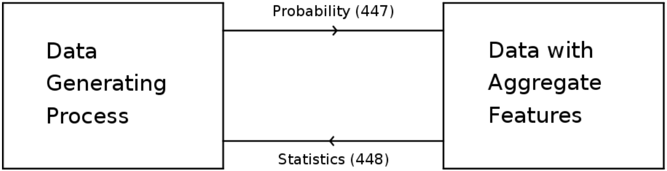

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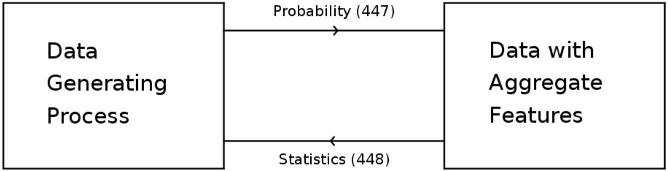

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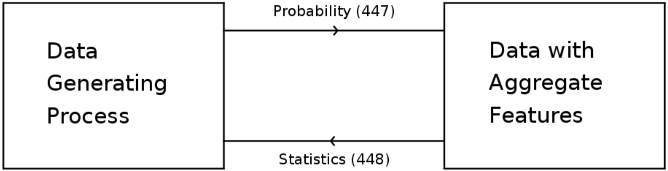

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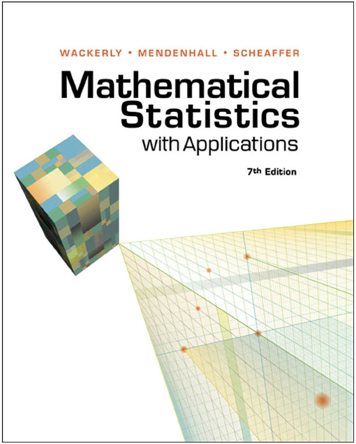

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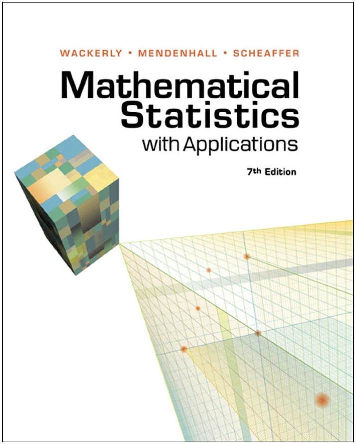

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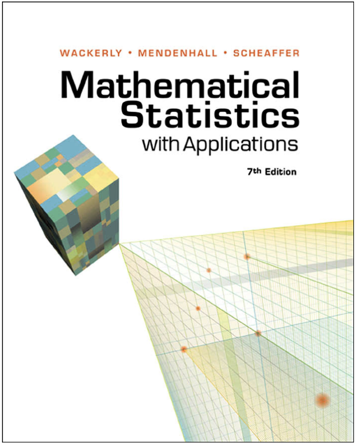

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Throughout the notes, the words "Text" and "Book" will refer to the book mentioned above.

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2. Bayes' Theorem Problem (e.g. Text 2.125)

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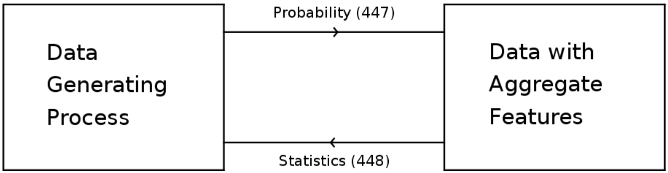

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6. Instructor's Solution Manual.

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- Engineering applicants: 90 men, 10 women
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End of Chapter 1

Chapter 2

Probability

An “Interview Problem”

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Example (3)

If the player flips T, H, T and then stops, their payout is $\frac{1}{3} = \$0.33\ldots$.

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- If we flip H on the first try, stop.
- If we flip T, flip 1000 times; the result will be very close to 50% heads, very likely.

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This is called the "Chow-Robbins Game". The exact value is unknown.

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- The fair price for the game is called an "Expected Value" or "Mean".

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On flipping two fair coins, the possible outcomes are HH, HT, TH, and TT, all equally likely. So the probability of each outcome is $\frac{1}{4} = 0.25$.

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Axiom 1: $P(A) \geq 0$.

Axiom 2: $P(S) = 1$.

Axiom 3: If $A_1, A_2, \dots$ form a sequence of pairwise mutually exclusive events in S (that is, $A_i \cap A_j = \emptyset$ if $i \neq j$), then

$$P(A_1 \cup A_2 \cup \dots) = \sum_{n=1}^{\infty} P(A_n).$$

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Note that this process only works when we know that all members of S are equally likely outcomes.

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5. Find $P(A)$ by summing the probabilities of the sample points in A.

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(c) What is the probability that the person chosen at random has either type A or type AB blood?

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Remark:

In a situation like this (not all simple events are equally likely), we need extra information to find the probabilities.

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In Section 2.8, we will see various probability formulas to get an idea of what's going on. Pretend we are in the situation of subsets of the unit square and that probability $=$ area.

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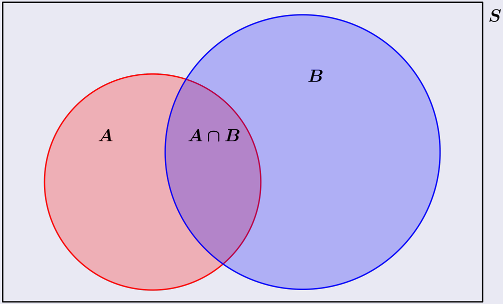

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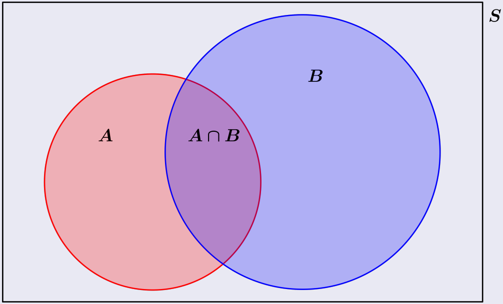

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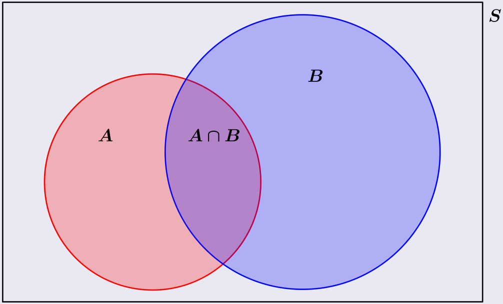

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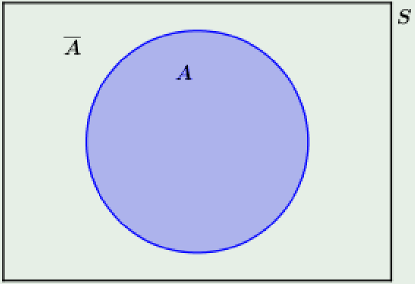

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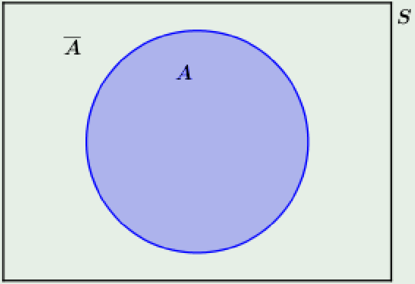

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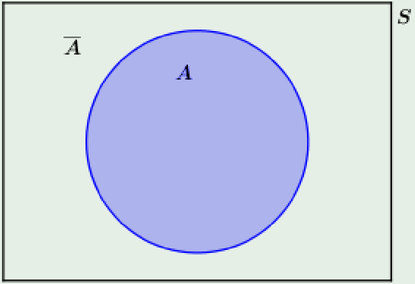

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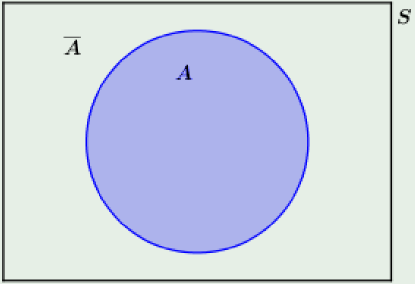

$$A \cup \overline{A} = S, A \cap \overline{A} = \emptyset \implies P(A) + P(\overline{A}) = P(S) = 1.$$

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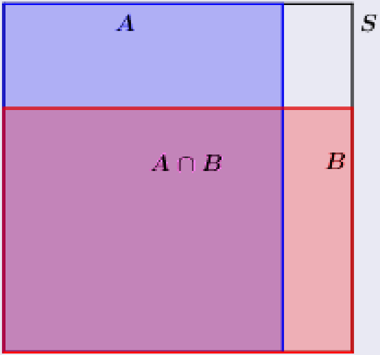

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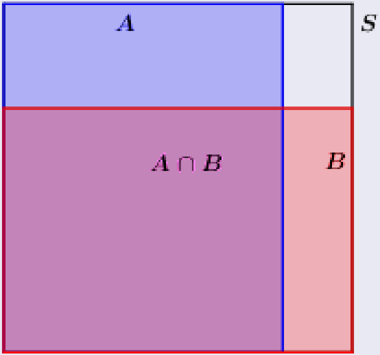

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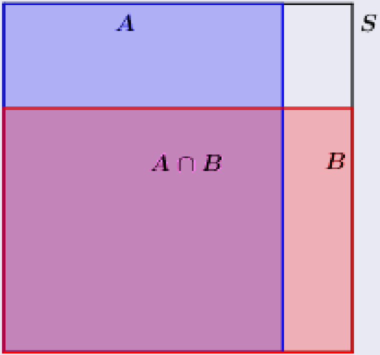

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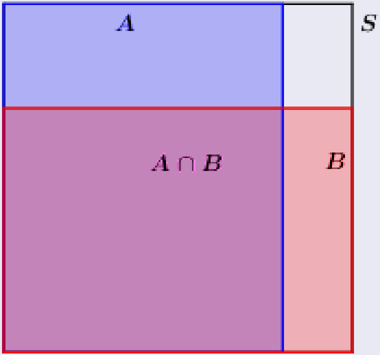

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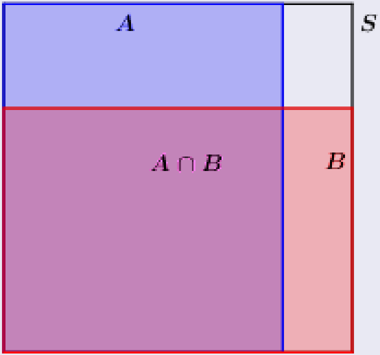

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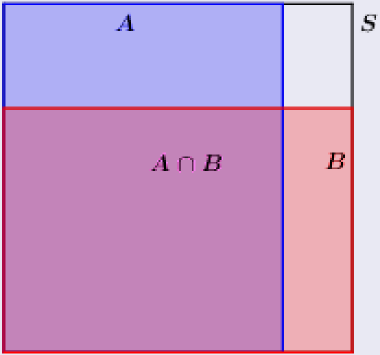

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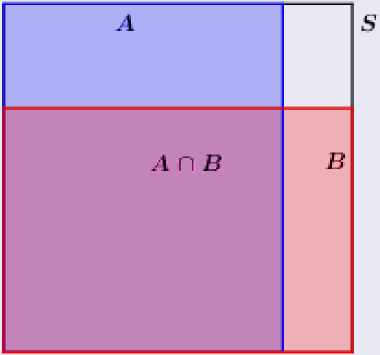

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(a) Determine the number of sample points in the sample space S for this experiment, that is, determine the number of ways the 20 laborers can be divided into groups of the appropriate sizes to fill all of the jobs.

(b) Find the probability of the observed event if it is assumed that the laborers are randomly assigned to jobs.

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How many ways can we do this so that all 4 members of the minority group are assigned to the most "unpleasant" job?

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- There are two possible mitigating factors:
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  - Maybe there were many chances to observe this event.

Recall:

Theorem (2.3)

If $n = n_1 + \cdots + n_k$, the number of ways of partitioning n objects into subsets of size $n_1, \ldots, n_k$ is the "Multinomial Coefficient"

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Remark: Binomial Coefficients

Binomial coefficients are a special case of Multinomial coefficients (k=2).

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If $n = n_1 + \cdots + n_k$, the number of ways of partitioning n objects into subsets of size $n_1, \ldots, n_k$ is the "Multinomial Coefficient"

$$\binom{n}{n_1 \ n_2 \ \ldots \ n_k} = \frac{n!}{n_1! \cdot n_2! \cdot \cdots \cdot n_k!}.$$

Remark: Binomial Coefficients

Binomial coefficients are a special case of Multinomial coefficients (k=2).

Recall the binomial coefficient is written $\binom{n}{k} = \dfrac{n!}{(n-k)! \cdot k!}$.

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Binomial Theorem:

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + y^n$$
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There is an analogous "Multinomial Theorem":

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Remark:

This is really saying $P(A) = \sum_{i=1}^{k} P(A \cap B_i)$.

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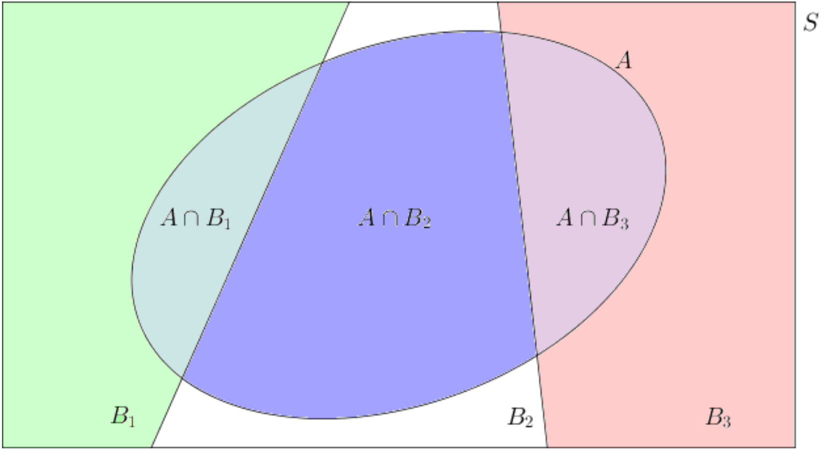

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Now apply the law of total probability in the denominator. $\square$

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<u>Interpretation:</u> There are two ways of testing positive:

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Conclusion:

The probability that the patient actually has the disease is only about <u>8%</u>.

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Analysis: What is the relative likelihood of these two events?

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Describe Vlad's optimal strategy in this 2-game match.

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Correct Strategy:

Play boldly in the first game. If win, play timidly in the second game. Otherwise, play boldly again.

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Suppose the probability that the student knows the answer to the question is .8 and the probability that the student will guess is .2.

Assume that if the student guesses, the probability of selecting the correct answer is .25.

If the student correctly answers a question, what is the probability that the student really knew the correct answer?

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Which strategy maximizes the c ontestant's probability of winning the good prize: stay with the initial choice or switch to the other curtain? In answering the following sequence of questions, you will discover that, perhaps surprisingly, this question can be answered by considering only the sample space above and using the probabilities that you assigned to answer part (a).

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Conclusion:

It is correct to switch curtains: the probability of winning by switching is $\frac{2}{3}$, while that of winning by not switching is $\frac{1}{3}$.

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What if there were 4 curtains instead of 3?

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We should switch because $\frac{3}{8} > \frac{1}{4}$.

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As $\$166.67 < \$200$, we should NOT switch for the cost.

End of Chapter 2

Chapter 3

Discrete Random Variables and Their Probability Distributions

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The average value of a random variable (over a large number of trials, say) is called the Expected Value of the random variable.

This is written $E[X]$ and we speak of "the expectation of X" or "the mean of X".

The formal definition captures some properties and subtleties not seen in our format. But this format is very convenient for computing the expected value $E[X]$ or "mean of X": in this case,

$$E[X] = 2 \cdot \frac{2}{3} + (-1) \cdot \frac{1}{3} = \frac{4}{3} - \frac{1}{3} = 1.$$

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Here X can take on values 2 and -1, and $P(X = 2) = {}^2/_3$, $P(X = -1) = {}^1/_3$. So the "probability function" $p(x) := P(X = x)$ is

$$p(x) = \begin{cases} {}^2/_3 & (x = 2) \\ {}^1/_3 & (x = -1) \\ 0 & \text{otherwise.} \end{cases}$$

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The expected value in switching is <u>less</u> than that without switching. So we should NOT switch.

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Solution: (continued)

To find $P(Y = 1)$ and $P(Y = 2)$, use the "Event-Composition Method" (recall Monty Hall: 5 curtains).

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Now translate the problem statement into probability statements about these events:

$$P(\overline{A} \cap \overline{B}) = 20\%, P(Y = 2) = P(A \cap B).$$

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y	$p(y) = P(Y = y)$
0	20%
1	70%
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We can now find $E[Y]$:

$$E[Y] = 0 \cdot 20\% + 1 \cdot 70\% + 2 \cdot 10\% = 0.7 + 0.2 = \boxed{0.9}.$$

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Observe that Y must be 2, 3, or 4.

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Solution: (continued)

$$\begin{array}{ccccc} \times & \times & \circ & \circ & (Y=2) \\ \times & \circ & \circ & \times & (Y=4) \\ \circ & \times & \circ & \times & (Y=4) \end{array} \qquad \begin{array}{ccccc} \times & \circ & \times & \circ & (Y=2) \\ \circ & \times & \times & \circ & (Y=3) \\ \circ & \circ & \times & \times & (Y=4) \end{array}$$

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In each of these cases, we can write down the number of the test on which the second defective is found. (Proceed left to right).

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We can use these properties to give a standard formula for “variance”.

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Note: We use one of the equivalent terms "distribution", "probability distribution", "probability function", and "probability mass function".

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To compute $E[X^2]$, we apply Theorem 3.2:

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Many of the variables of Chapter 3 are built from repeated independent Bernoulli trials. Example: Binomial random variable with parameters n and p.

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Equivalently, Y is Binomial with parameters n and p if Y has the probability function

$$p(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & (k = 0, 1, \ldots, n), \\ 0 & \text{otherwise.} \end{cases}$$

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Why must Y have the above probability function?

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Terminology: Sometimes we say "Y has the binomial distribution with parameters n and p".

If we have k successes in n trials, there are $\binom{n}{k}$ ways to distribute these.

Once we consider a particular pattern of successes, each success has probability p and each failure has it $1 - p$.

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We can now compute $E[Y]$ and $V[Y]$.

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Alternate derivation is given in Theorem 3.7.

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Do the computations to obtain $V[Y] = npq$.

$\square$

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From the description above, we can find the probability function, mean, and variance of Y.

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From the above probability function, we can write down an infinite series for $E[Y]$:

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(This is an exercise in Calculus 2.)

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Conclusion: $E[Y] = \dfrac{1}{p}$.

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Exercise:

Do the work outlined above to get

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The probability of (1) is just p, and (1) and (2) are independent.

Put this all together to get

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4. The waiting time for each success is a geometric random variable with parameter p. This means that the negative binomial random variable with parameters r and p is the sum of r independent geometric random variables, each with parameter p.

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So if we remember that for a geometric RV the mean is $\frac{1}{p}$ and the variance is $\frac{q}{p^2}$, then the mean and variance of a negative binomial RV are easy to remember.

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Given that we have already tossed a balanced coin ten times and obtained zero heads, what is the probability that we must toss it at least two more times to obtain the first head?

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- The word "balanced" in the problem statement means that this probability is $\frac{1}{2}$.

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We have a bag of 100 coins. One is "double-tails" and 99 are normal. We pick one coin from the bag and flip it 10 times. It comes up tails 10 times in a row. What are the chances that it is actually the trick coin?

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Compute with Bayes' Rule.

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Let's try now to produce a table with the probability function, mean, and variance for all of these RVs.

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Binomial	$\binom{n}{y} p^y q^{n-y}$	np	npq	
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This convention is relevant in evaluating the probability function for the hypergeometric distribution.

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Hypergeometric	$\dfrac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}}$	$\frac{nr}{N}$	$\frac{nr}{N} \cdot \frac{N-r}{N} \cdot \frac{N-n}{N-1}$

$[q = 1 - p]$

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Easy to remember, but the derivation requires some work with power series:

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The derivation of $V[Y]$ involves similar tricks.

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Note:

This result is important in understanding problems which say "Y is a Poisson distributed with average rate ____".

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Remark:

We do this because the distribution is determined by the moment generating function (MGF).

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So if $Y \sim \text{Bin}(n, p)$, then $\boxed{m_Y(t) = (q + pe^t)^n}$.

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Where did we do this? Note that an infinite sum is a limit:

$$\sum_{n=1}^{\infty} a_n \overset{\text{def}}{=} \lim_{N\to\infty} \left(\sum_{n=1}^{N} a_n \right).$$

A derivative is a limit:

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Using the connection between the geometric and the negative binomial RVs, namely that the negative binomial RV is the sum of r independent geometric RVs, we get, for the negative binomial RV,

$$m(t) = \left(\frac{pe^t}{1-qe^t}\right)^r.$$

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$$A = \{Y = 1, 3, 5, \ldots \text{(an odd number)}\}.$$

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Solution: (continued)

Now plug in $p = {}^1/_6, q = {}^5/_6$ and find

$$P(B) = \frac{1}{6}\frac{5}{6} \cdot \frac{1}{1 - {}^{25}/_{36}}$$

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$$P(B \text{ wins on turn } 2) = \begin{array}{l} P(A \text{ doesn't win on turn } 1) \cdot \\ \quad P(B \text{ wins at } 2 \mid A \text{ doesn't win at } 1) \end{array} = \frac{5}{6}\frac{1}{6},$$

and similarly for the rest of the terms.

Solution: (continued)

Now plug in $p = 1/6, q = 5/6$ and find

$$P(B) = \frac{1}{6}\frac{5}{6} \cdot \frac{1}{1 - 25/36} = \frac{5}{36}\frac{1}{(11/36)} = \frac{5}{11}.$$

Other ways to think about this:

$P(B) = 1 - P(A)$, so we can compute $P(A)$ instead.

Also, instead of using what we know about geometric RVs, we could write down directly the probability that B wins on turn $2 + B$ wins on turn $4 + \ldots$:

$$P(B \text{ wins on turn } 2) = \begin{array}{l} P(A \text{ doesn't win on turn } 1) \cdot \\ \quad P(B \text{ wins at } 2 \mid A \text{ doesn't win at } 1) \end{array} = \frac{5}{6}\frac{1}{6},$$

and similarly for the rest of the terms.

This is basically re-deriving the probability function for the geometric RV.

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Consequence: $m_Y^{(k)}(0) = \mu'_k = E[Y^k]$.

In particular, $m_Y^{(1)}(0) = \mu_1' = E[Y]$.

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Caution: $V[Y]$ is not $m_Y^{(2)}(0)$, unless $E[Y] = 0$.

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The result says that $P(Y \text{ is far from mean})$ is small.

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$$P(|Y-\mu| \geq k\sigma) \leq \frac{1}{k^2}, \quad P(|Y-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}.$$

Note that the two events are complementary: if $|Y-\mu|$ is not $\geq k\sigma$, then $|Y-\mu| < k\sigma$.

What does this really say?

Let's set $k = 3$. Then the first statement says

$$P(|Y-\mu| \geq 3\sigma) \leq \frac{1}{3^2} = \frac{1}{9}.$$

The result says that $P(Y$ is far from mean$)$ is small. In particular, the probability that Y is 3 or more standard deviations away from its mean is less than or equal to $\frac{1}{9}$.

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4. The bound is weak, in the sense that it can be greatly improved with the knowlegde of the distribution.

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Remark:

We specialized $k = 3$ for clarity. But a similar example can be constructed with $\frac{1}{18}$ replaced by $\frac{1}{2k^2}$ for any k, and get similar results.

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Notice that we had to know the probability function for the Poisson RV.

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Interpretation:

We are looking at the expectation of a function of a geometric RV.

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3. 4 points are chosen at random on the unit sphere in $\mathbb{R}^3$. They form a tetrahedron. What is the probability that the origin (the center of the sphere) lies in the tetrahedron?

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End of Chapter 3

Chapter 4

Continuous Variables and Their Probability Distributions

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First note that if $y < 0$, then $P(Y \leq y) = 0$.

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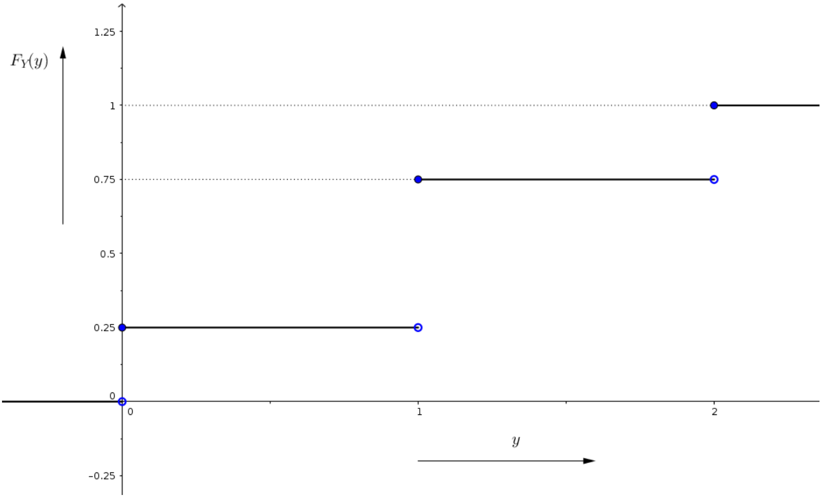
Graph of $F_Y(y)$:



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Define

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And this is what we will do for most of Chapter 4. Note by the Fundamental Theorem of Calculus,

$$f_Y(y) = \frac{d}{dy} F_Y(y).$$

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Recall that we studied $F(Y)$ where $Y \sim \text{Bin}(2, \frac{1}{2})$.

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Notice that $F_Y : \mathbb{R} \to \mathbb{R}$. This is different from the RV Y in the sense that the domain of Y is some "sample space", i.e. $Y : S \to \mathbb{R}$.

Properties of a distribution function:

(1)

$$\lim_{y \to -\infty} F_Y(y) = 0. \qquad \text{[Book writes } F_Y(-\infty) = 0\text{]}$$

(2)

$$\lim_{y \to \infty} F_Y(y) = 1. \qquad \text{[Book writes } F_Y(\infty) = 1\text{]}$$

(3) F_Y is nondecreasing: if $y_1 < y_2$, then $F_Y(y_1) \leq F_Y(y_2)$.

(4) F_Y is "right continuous".

Recall that we studied $F(Y)$ where $Y \sim \text{Bin}(2, \frac{1}{2})$. We saw that F_Y is a step function.

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ 1/4 & 0 \le y < 1 \\ 3/4 & 1 \le y < 2 \\ 1 & y \ge 2. \end{cases}$$

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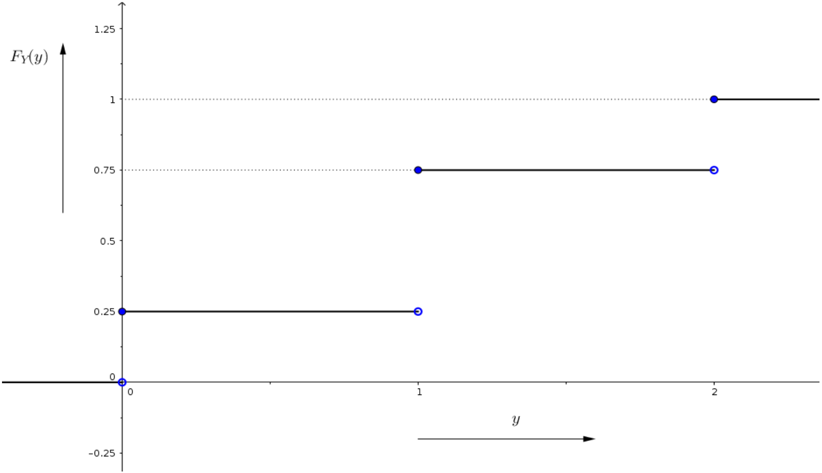

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In this book we can just assume that F_Y is differentiable.

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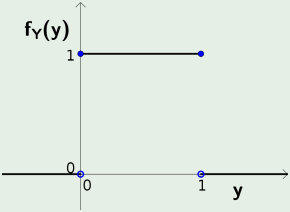

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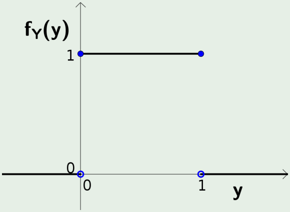

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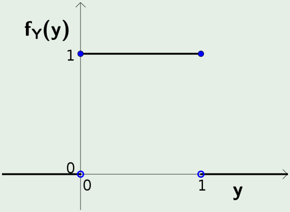

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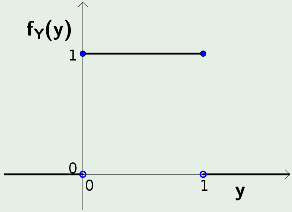

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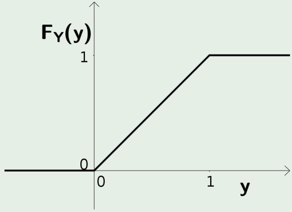

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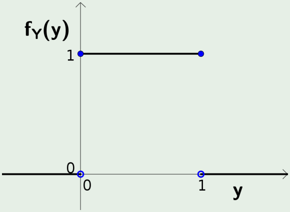

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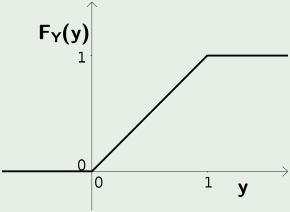

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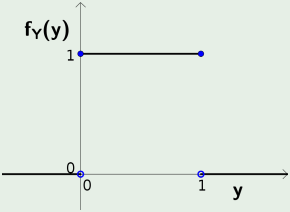

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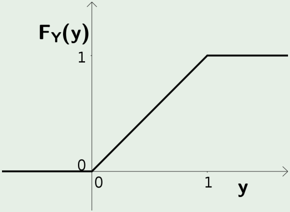

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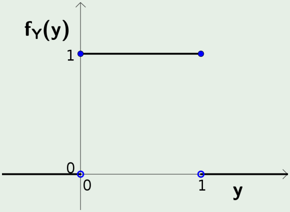

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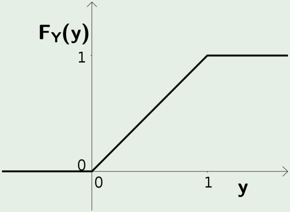

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We just defined a continuous RV Y. This Y is said to have "the uniform distribution" on the interval $[0, 1]$.

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Suppose that Y possesses the density function

$$f(y) = \begin{cases} cy & 0 \leq y \leq 2 \\ 0 & \text{elsewhere.} \end{cases}$$

Find the value of c that makes $f(y)$ a probability density function.

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In various exercises you may also see Weibull distribution, Pareto distribution, Rayleigh distribution, and some more.

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If Y is normal with parameters μ and σ (denoted $Y \sim \mathcal{N}(\mu, \sigma^2)$), then $E[Y] = \mu$ and $V[Y] = \sigma^2$.

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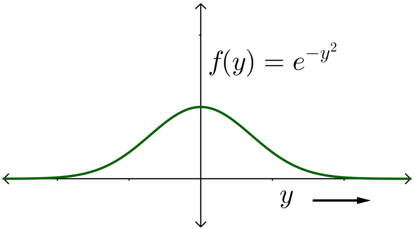

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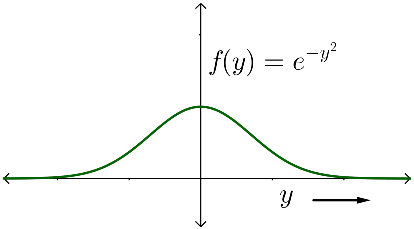

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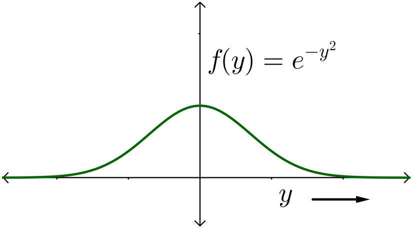

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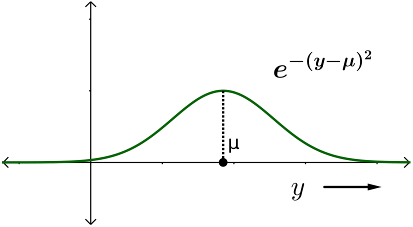
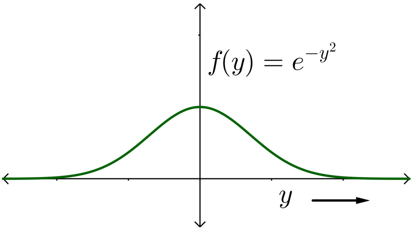


Remarks on the function:

Start with e^{-y^2}.

This is a "Bell Curve".



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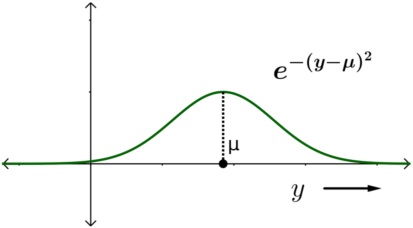

What is

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Our PDF needs to be

$$f(y) = \frac{1}{\sqrt{\pi}} e^{-y^2},$$

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When we are done with all this "adjusting" and "normalizing", we get

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}.$$

If Y is a normal RV with mean μ and variance σ^2, then

$$P(a \le Y \le b) = \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}\,dy.$$

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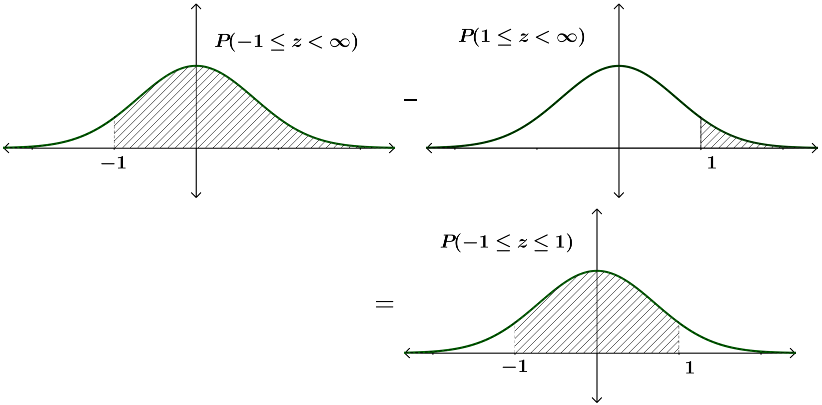
Two tricks:

1. If Y is normal with mean μ and standard deviation σ (variance σ^2), then $\dfrac{Y-\mu}{\sigma}$ is <u>also normal</u> with mean 0 and variance 1.
2. Suppose, for a standard normal RV Z, we want $P(-1 \leq z \leq 1)$. This is the same as $P(-1 \leq z < \infty) - P(1 \leq z < \infty)$: (pictures follow:)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-1
$-$
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1
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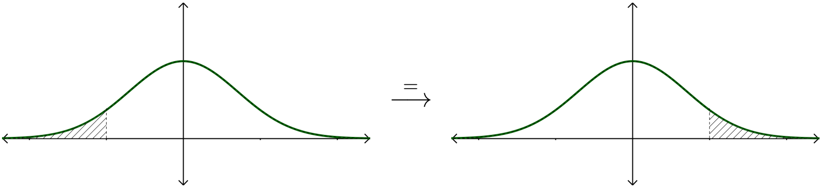

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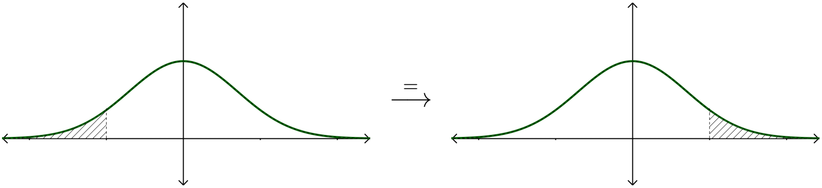

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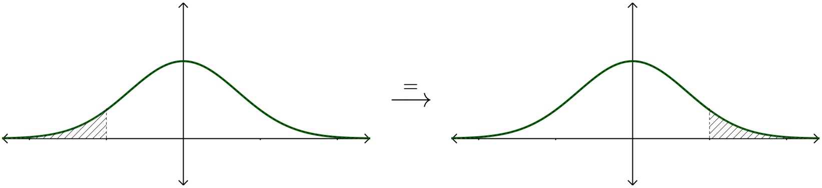

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Noting that $Z = \dfrac{Y-75}{10}$ is standard normal, we need to find $P(0.5 \leq Z \leq 1.5)$. This is

$$\int_{0.5}^{1.5} f_Z(y)dy = \int_{0.5}^{\infty} f_Z(y)dy - \int_{1.5}^{\infty} f_Z(y)dy.$$

where f_Z is the standard normal PDF.

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- Warning: Make sure you know what kind of table you have.

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Exercise 4.73(a):

The width of bolts of fabric is normally distributed with mean 950 mm (millimeters) and standard deviation 10 mm. What is the probability that a randomly chosen bolt has a width of between 947 and 958 mm?

Remarks: (continued)

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Interpretation:

Find $P(947 \leq Y \leq 958)$, where $Y \sim \mathcal{N}(950, 100)$.

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We have

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Remark:

The book does not use the erfc notation.

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In practical problems, it may be appropriate to use a different distribution, if we are interested in, say, $P(Z \geq 5)$.

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Definition (The Gamma Distribution)

Y has the Gamma distribution with parameters α and β if the PDF is

$$f_Y(y) = \begin{cases} \dfrac{y^{\alpha-1}e^{-\frac{y}{\beta}}}{\beta^\alpha \Gamma(\alpha)} & y \geq 0, \\ 0 & y < 0, \end{cases}$$

where

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}\,dy$$

is the "generalized factorial function".

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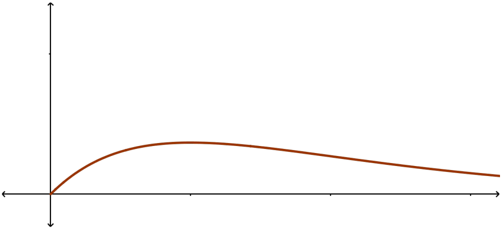

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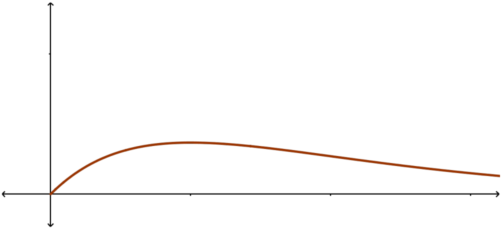

Note that the number $\beta^\alpha \Gamma(\alpha)$ is put in the PDF so that

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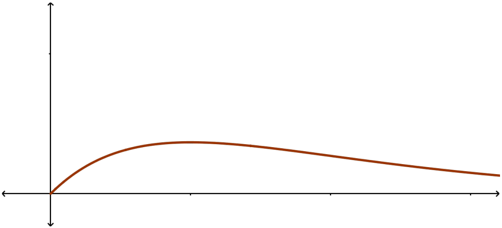

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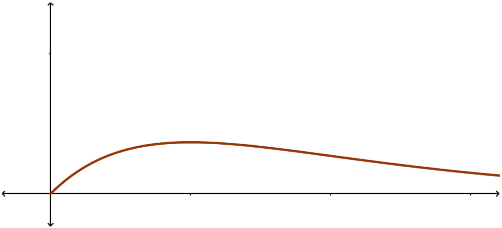

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$$dv = \frac{dy}{\beta}, dy = \beta dv, y = \beta v.$$

Using the change-of-variable, we get

$$\int_0^{\infty} \frac{y^{\alpha-1}e^{-\frac{y}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)}dy = \int_{v=0}^{v=\infty} \frac{(\beta v)^{\alpha-1}e^{-v}}{\beta^{\alpha}\Gamma(\alpha)}\beta dv$$

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If $Y \sim \Gamma(\alpha, \beta)$, then $E[Y] = \alpha\beta$ and $V[Y] = \alpha\beta^2$.

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Exercises 4.194 and 4.196

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Substitute $v = \dfrac{y}{\beta}$, so that $y = \beta v$ and $dy = \beta dv$.

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Finding $V[Y]$ is similar, but much more complicated.

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The reason we study this separately is in Theorem 7.2: If $Z_1, \ldots, Z_k$ are independent standard normal RVs and $Y = Z_1^2 + \cdots + Z_k^2$ (think of Y as a sum of squared errors), then $Y \sim \chi^2[k]$.

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Exercise 4.89 (a):

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All of this is based on integration by parts, it's just made easier with clever packaging.

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Suppose that Y has a Gamma distribution with parameters α and β. If a is any positive or negative value such that $\alpha + a > 0$, show that

$$E[Y^a] = \frac{\beta^a \Gamma(\alpha + a)}{\Gamma(\alpha)}.$$

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Remark:

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This, like so much else, is proved by integration by parts.

Exercise 4.125:

The percentage of impurities per batch in a chemical product is a random variable Y with density function

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Sometimes people think in terms of "central moments" $\mu_k = E[(Y-\mu)^k]$, but we can derive these from the μ'_k (and vice versa).

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Now rearrange this to get an integral we can evaluate using the standard formula for Γ integrals.

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So if Y is normal with mean 0 and variance σ^2, and $X = Y + \mu$, then

$$m_X(t) = e^{\mu t}m_Y(t)$$

Further trickery in integration will show that for a normal RV with mean 0 and variance σ^2 that $m(t) = e^{\left(\frac{\sigma^2 t^2}{2}\right)}$. From this, we can derive the MGF of any normal RV.

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Exercise in "trickery": (Example 4.16)

Find the MGF for $g(Y) = Y - \mu$, where $Y \sim \mathcal{N}(\mu, \sigma^2)$.

Exercise 4.139

If $Y \sim \mathcal{N}(\mu, \sigma^2)$ and $X = -3Y + 4$, find $m_X(t)$.

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What is the distribution of X?

X is normal with mean $-3\mu + 4$ and variance $(3\sigma)^2$ (or standard deviation 3σ), because X has the same MGF as a normal RV mean $-3\mu + 4$ and standard deviation 3σ.

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If Y is any RV with mean μ and standard deviation σ, then

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&\geq k^2\sigma^2 \cdot P(|Y-\mu| \geq k\sigma) \implies P(|Y-\mu| \geq k\sigma) \leq \frac{1}{k^2}.
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When would you use Tchebysheff's theorem? Mainly when you don't know the distribution of the RV being studied.

Exercise 4.147:

A machine used to fill cereal boxes dispenses, on the average, μ ounces per box. The manufacturer wants the actual ounces dispensed Y to be within 1 ounce of μ at least 75% of the time. What is the largest value of σ, the standard deviation of Y, that can be tolerated if the manufacturer's objectives are to be met?

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we find $k = 2$ and $\sigma = \frac{1}{2}$.

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- If you want to go on, consider doing some of these exercises.

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According to the model, heights between 5'9" and 6'0' have z-scores between 0 and 1. According to the $68\% - 95\% - 99.7\%$ rule, and symmetry, this gives $34\% \times 110$ million ≈ 37.4 million.

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According to the model, heights between 5'9" and 6'0' have z-scores between 0 and 1. According to the $68\% - 95\% - 99.7\%$ rule, and symmetry, this gives $34\% \times 110$ million ≈ 37.4 million.

(This could be improved with better data from the Census Bureau.)

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Question: Who counts as American? (Green Card? Only Citizens? etc.)

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Sometimes the model isn't appropriate for answering the question being asked – question (b) is one of those times.

End of Chapter 4

Chapter 5

Multivariate Probability Distributions

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1. $p(y_1, y_2) \geq 0$.
2. $$\sum_{y_1} \sum_{y_2} p(y_1, y_2) = 1.$$

Example 5.1:

A local supermarket has three checkout counters. Two customers arrive at the counters at different times when the counters are serving no other customers. Each customer chooses a counter at random, independently of the other. Let Y_1 denote the number of customers who choose counter 1 and Y_2, the number who select counter 2. Find the joint probability function of Y_1 and Y_2.

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y_2 \ y_1	0	1	2
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The decisions of the customers are independent, so the probability is $2/3 \times 2/3 = 4/9$.

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From the joint probability function, we can derive the individual probability functions ("Marginal Probability Functions"):

$$p_1(y_1) = \sum_{\text{all } y_2} p(y_1, y_2).$$

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So the terminology "marginal distribution" comes from the fact that this is the distribution on the margin of the original table.

$y_2 \backslash y_1$	0	1	2	$p_2(y_2) \downarrow$
0	$1/9$	$2/9$	$1/9$	$4/9$
1	$2/9$	$2/9$	0	$4/9$
2	$1/9$	0	0	$1/9$
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All of this is also treated for continuous RVs.

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All of this is also treated for continuous RVs. Recall that when working with continuous RVs, we introduced the "cumulative distribution function" F_Y: $F_Y(y) = P(Y \leq y)$.

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1	$2/9$	$2/9$	0	$4/9$
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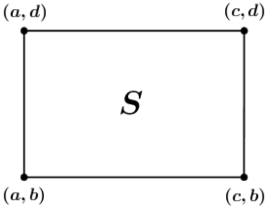

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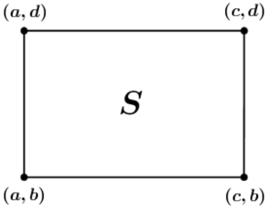

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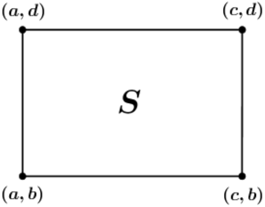

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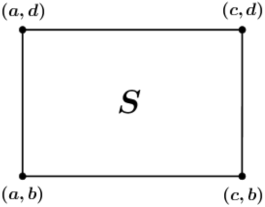

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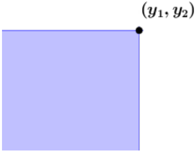
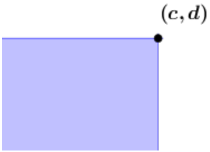
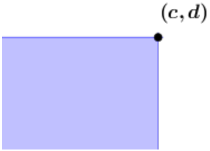


$X =$ 

$X =$ 

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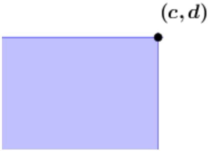
$X =$ 

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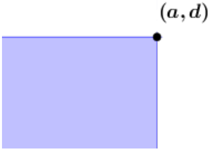
$A =$ 

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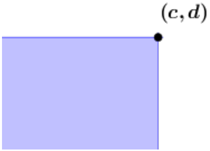

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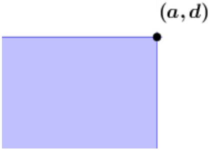

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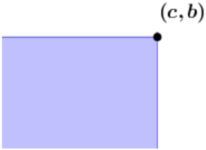
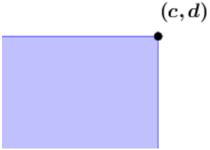


$X =$ 

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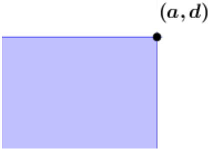
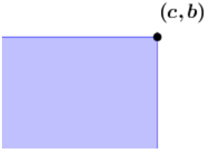
$A =$ 

$B =$ 

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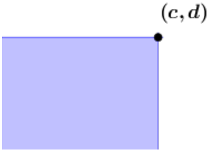

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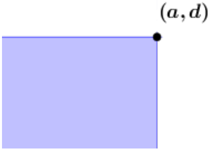

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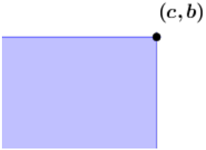

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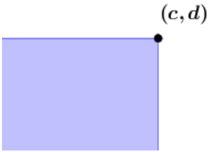
$X =$ 

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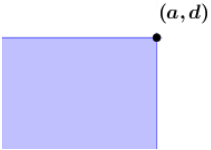
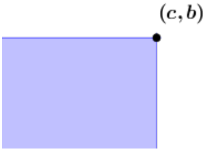
$A =$ 

$B =$ 

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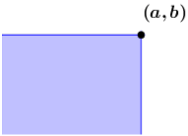
$A \cap B =$ 

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Note that this JDF has other properties which are analogous to the properties of a distribution function for a single RV, e.g.
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$$F(c,d) - F(a,d) - F(c,b) + F(a,b) \geq 0$$
whenever $d \geq b$ and $c \geq a$, because $P((Y_1, Y_2) \in S) \geq 0$.

Note that this JDF has other properties which are analogous to the properties of a distribution function for a single RV, e.g.
$$\lim_{y_1 \to \infty} \lim_{y_2 \to \infty} F(y_1, y_2) = 1. \qquad \text{(see p.228 in the text.)}$$

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Define the joint distribution of two RVs Y_1, Y_2 by taking them to be the coordinates of a point chosen at random from the unit square $[0,1] \times [0,1]$. Find $P(0.1 \leq Y_1 \leq 0.3, 0 \leq Y_2 \leq 0.5)$.

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We must find the joint density function $f(y_1, y_2)$.

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We must find the joint density function $f(y_1, y_2)$. "Chosen at random" means "uniform distribution", which in turn implies that "The density function is constant in some region and is 0 elsewhere".

Here the region is the unit square $[0,1] \times [0,1]$.

Solution: (continued)

The density function is required to satisfy

$$\text{Total Probability } = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1\, dy_2 = 1.$$

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Or, more simply,

$$\int_0^1\int_0^1 c\, dy_1\, dy_2 = \text{area of unit square} \cdot c = c.$$

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Remark:

The double integrals and setup can get more complicated; not every region is a rectangle.

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Remark:

In the definition of density functions, do not forget about the "$f = 0$ elsewhere" clause:

$$\int_{-\infty}^{0.2} 1\, dy_1$$

does not converge.

Example 5.4:

Gasoline is to be stocked in a bulk tank once at the beginning of each week and then sold to individual customers. Let Y_1 denote the proportion of the capacity of the bulk tank that is available after the tank is stocked at the beginning of the week. Because of the limited supplies, Y_1 varies from week to week. Let Y_2 denote the proportion of the capacity of the bulk tank that is sold during the week. Because Y_1 and Y_2 are both proportions, both variables take on values between 0 and 1. Further, the amount sold, y_2, cannot exceed the amount available, y_1. Suppose that the joint density function for Y_1 and Y_2 is given by

$$f(y_1, y_2) = \begin{cases} 3y_1 & 0 \leq y_2 \leq y_1 \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Find the probability that less than one-half of the tank will be stocked and more than one-quarter of the tank will be sold.

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We convert the above paragraph into some simpler-looking math:

Interpretation:

Suppose Y_1 and Y_2 have the joint density function

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Find $P(0 \leq Y_1 \leq 0.5, Y_2 > 0.25)$.

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Start by graphing the region of integration.

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Where is $f(y_1, y_2)$ nonzero?

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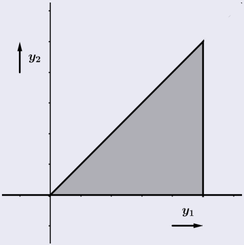

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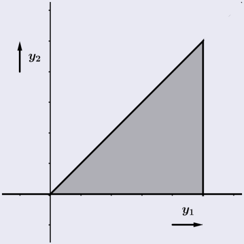

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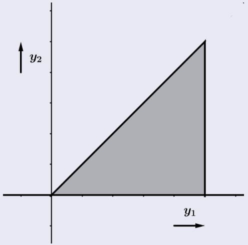

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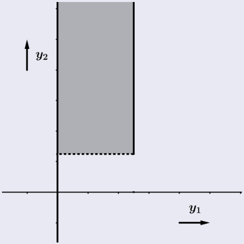

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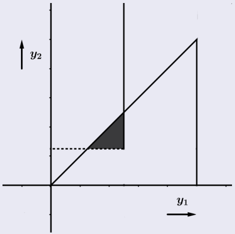

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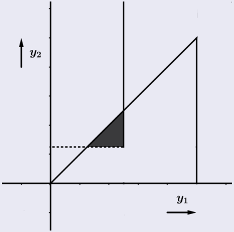

$$P(0 \leq Y_1 \leq 0.5, Y_2 > 0.25) = \int_0^{0.5} \int_{0.25}^{\infty} f(y_1, y_2)\, dy_2\, dy_1.$$

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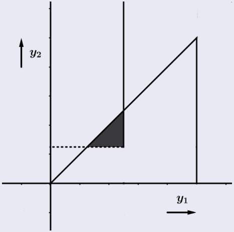

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But, since $f = 0$ outside the small triangle, this is

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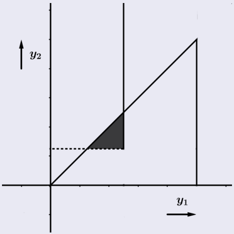

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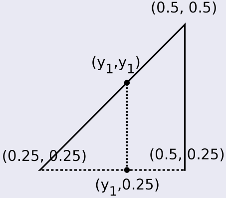
$$\int_0^{0.5} \int_{0.25}^{y_1} (3y_1)\, dy_2\, dy_1.$$

Close-up of the small triangle:



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For an example where you must find the density function, see problems 5.8-5.11 in Section 5.2.

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Remark:

Remembering these definitions is essential to being able to do problems without the aid of the text.

Exercise 5.33 (a):

Suppose that Y_1 is the total time between a customer's arrival in the store and departure from the service window, Y_2 is the time spent in line before reaching the window, and the joint density of these variables is

$$f(y_1, y_2) = \begin{cases} e^{-y_1} & 0 \leq y_2 \leq y_1 < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

Find the marginal density functions for Y_1 and Y_2.

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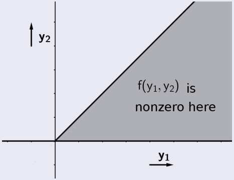

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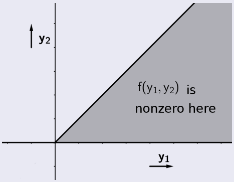

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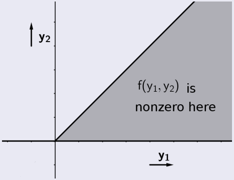

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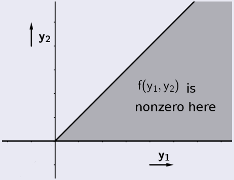

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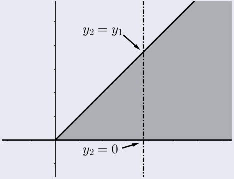

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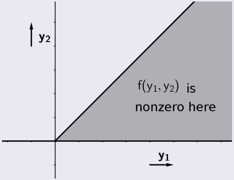

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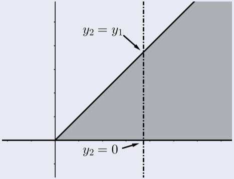

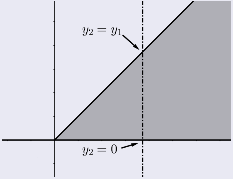

If $y_2 < 0$, we integrate over a line, and $f(y_1, y_2)$ is identically zero on this line.

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$$f(x)g(x) = \begin{cases} x \cdot x & x \geq 0, -1 \leq x \leq 1 \\ x \cdot 0 & x \geq 0, x \notin [-1,1] \\ -x \cdot x & x < 0, -1 \leq x \leq 1 \\ -x \cdot 0 & x < 0, x \notin [-1,1]. \end{cases}$$

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$$f(x)g(x) = \begin{cases} x \cdot x & x \geq 0, -1 \leq x \leq 1 \\ x \cdot 0 & x \geq 0, x \notin [-1, 1] \\ -x \cdot x & x < 0, -1 \leq x \leq 1 \\ -x \cdot 0 & x < 0, x \notin [-1, 1]. \end{cases}$$

Thus

$$f(x)g(x) = \begin{cases} x^2 & x \in [0, 1] \\ -x^2 & x \in [-1, 0) \\ 0 & x \notin [-1, 1]. \end{cases}$$

Remark on multiplication / division of case-defined functions:

If f has 2 cases and g has 2 cases, then fg has 4 cases.

Example:

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What we did for 5.33 (b) was much like this.

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An important property of independent RVs:

If Y_1, Y_2 are independent, then $E[Y_1 Y_2] = E[Y_1]E[Y_2]$.

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Expectation has the same linearity properties we studied before. These can be used to simplify many problems.

Exercise 5.74:

Suppose that a radioactive particle is randomly located in a square with sides of unit length. A reasonable model for the joint density function for Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} 1 & 0 \le y_1 \le 1, 0 \le y_2 \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Find (a) $E[Y_1 - Y_2]$. (b) $E[Y_1 Y_2]$. (c) $E[Y_1^2 + Y_2^2]$. (d) $V[Y_1 Y_2]$.

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without any integration at all.

Tricks for Double-Integral Problems

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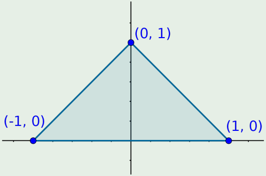
The joint distribution of Y_1, Y_2 is uniform over the triangle shown alongside. Find $E[Y_1]$.



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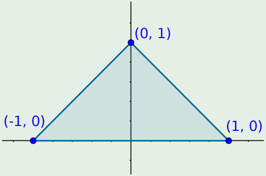

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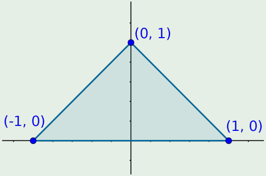

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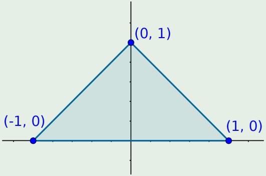

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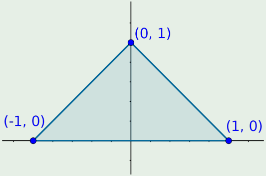

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where f is the joint density function. What is this joint density function? "Uniform" means f is a constant in the triangle (let's call it T) and 0 outside.

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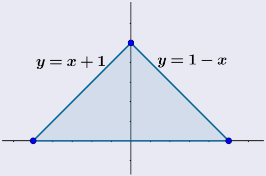

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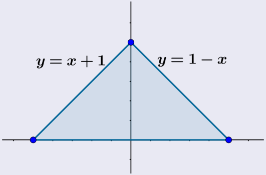

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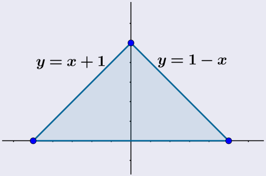

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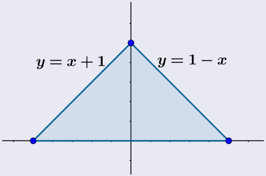

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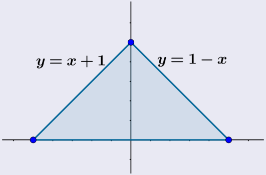

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Solution by formal procedure and integration: (continued)

$$= \left(\frac{y^3}{3} + \frac{y^2}{2} \right) \Bigg|_{-1}^{0} + \left(\frac{y^2}{2} - \frac{y^3}{3} \right) \Bigg|_{0}^{1}$$

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Solution by symmetry:

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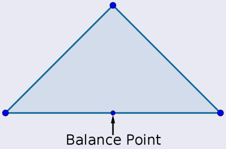

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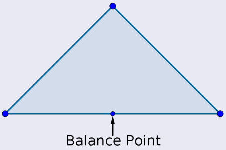

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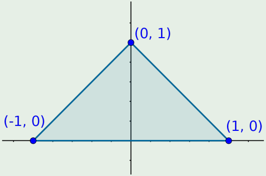

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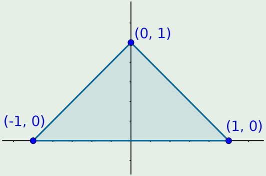

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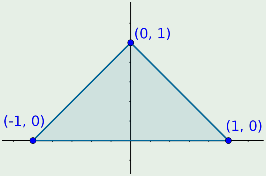

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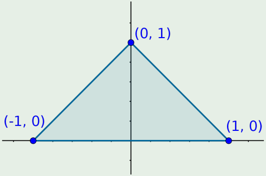

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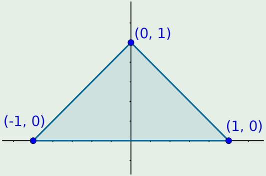

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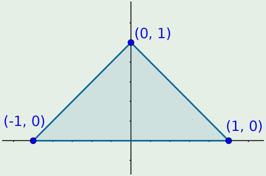

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$$E[Y_2] = \iint_T y_2 \, dy_1 \, dy_2 \qquad \text{(just like before)}$$
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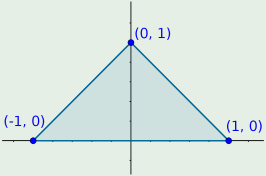

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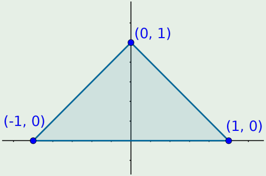

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Solution: (continued)

$$= \int_{-1}^{0} \left(\frac{(y_1^2}{2} + y_1 + \frac{1}{2} \right) dy_1 + \int_{0}^{1} \left(\frac{1}{2} - y_1 + \frac{y_1^2}{2} \right) dy_1$$

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$$= \left(\frac{y_1^3}{6} + \frac{y_1^2}{2} + \frac{y_1}{2} \right) \Big|_{-1}^{0} + \left(\frac{y_1}{2} - \frac{y_1^2}{2} + \frac{y_1^3}{6} \right) \Big|_{0}^{1}$$

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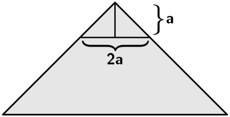

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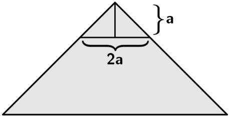

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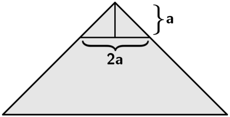

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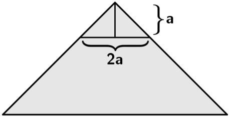

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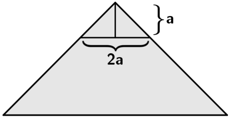

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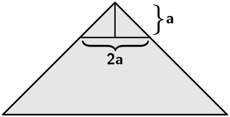

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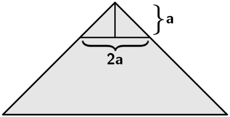

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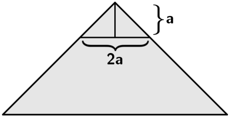

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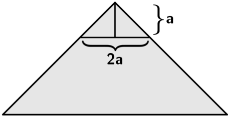

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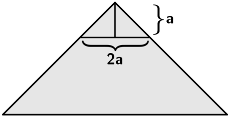

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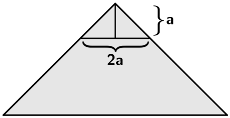

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In order to compute these, we need to know the joint distribution of Y_1 and Y_2.

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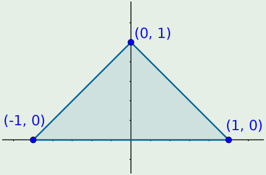

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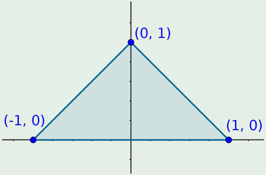

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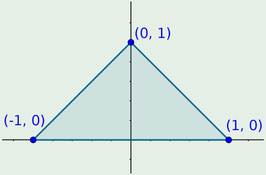

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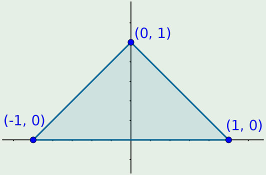

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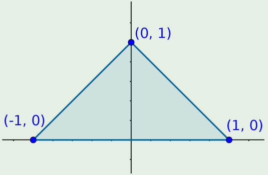

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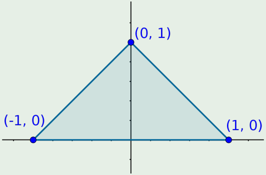

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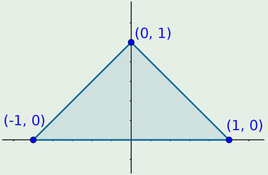

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So $\mathrm{Cov}(Y_1, Y_2) = 0$, but Y_1 and Y_2 are NOT independent.

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The covariance is a measure of the extent to which X and Y "vary together". Note that the covariance can be negative.

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Let $Y_1, \ldots, Y_n$ and $X_1, \ldots, X_m$ be random variables with $E(Y_i) = \mu_i$ and $E(X_j) = \nu_j$. Define

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The above theorem presents an important property of covariance: Covariance is bilinear, that is, it is a function of two variables which is separately linear in each variable.

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How does this give us the complicated statement of Theorem 5.12?

Observe that $V[X] = \text{Cov}(X, X)$.

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&\quad + \qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
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&= \sum_{j=1}^{m} b_j^2 V[X_j] + 2 \sum\sum_{1 \le i < j \le n} b_i b_j \operatorname{Cov}(X_i, X_j).
\end{aligned}$$

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Exercise 5.112

Let Y_1 and Y_2 denote the lengths of life, in hundreds of hours, for components of types I and II, respectively, in an electronic system. The joint density of Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} \frac{y_1}{8} e^{-(y_1 + y_2)/2} & y_1 > 0, y_2 > 0, \\ 0 & \text{elsewhere.} \end{cases}$$

The cost C of replacing the two components depends upon their length of life at failure and is given by $C = 50 + 2Y_1 + 4Y_2$. Find $E[C]$ and $V[C]$.

Solution:

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Additive constants don't matter for variance and covariance, i.e. $V[X + a] = V[X]$. Also $\mathrm{Cov}(X + a, Y) = \mathrm{Cov}(X, Y)$.

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$$\begin{aligned} &\text{Cov}(2Y_1 + 4Y_2, 2Y_1 + 4Y_2) \\ = &\, 2\text{Cov}(Y_1, 2Y_1 + 4Y_2) + 4\text{Cov}(Y_2, 2Y_1 + 4Y_2) \end{aligned}$$

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&= 2\text{Cov}(Y_1, 2Y_1 + 4Y_2) + 4\text{Cov}(Y_2, 2Y_1 + 4Y_2) \\
&= 2 \cdot 2\text{Cov}(Y_1, Y_1) + 4 \cdot 2\text{Cov}(Y_1, Y_2) + 2 \cdot 4\text{Cov}(Y_2, Y_1) \\
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=\ & 2 \cdot 2\mathrm{Cov}(Y_1, Y_1) + 4 \cdot 2\mathrm{Cov}(Y_1, Y_2) + 2 \cdot 4\mathrm{Cov}(Y_2, Y_1) \\
& + 4 \cdot 4\mathrm{Cov}(Y_2, Y_2) \qquad [\text{Remark: } \mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)]
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Remarks:

- This dupliates the calculation for Theorem 5.12, but with 4 terms and not m^2 terms.
- The biggest "difficulty" in some of the problems is setting up and doing double integrals.

Practice Problem 1: Exercise 5.27 (b)

Given that the joint density function of Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} 6(1 - y_2) & 0 \le y_1 \le y_2 \le 1, \\ 0 & \text{elsewhere,} \end{cases}$$

find $P\left(Y_2 \le {}^{1}\!/\!{}_{2} \mid Y_1 \le {}^{3}\!/\!{}_{4}\right)$.

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We need to know $\dfrac{P\left(Y_2 \le 1/2, Y_1 \le 3/4\right)}{P\left(Y_1 \le 3/4\right)}$.

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First draw the region where $f \neq 0$:

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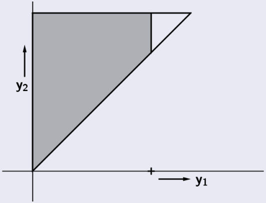

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Solution: (continued)

$$P\left(Y_1 \leq \frac{3}{4}\right) = \iint\limits_{\text{shaded region}} 6(1-y_2)\,dy_1\,dy_2$$

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Solution: (continued)

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\begin{aligned}
P\left(Y_1 \le \frac{3}{4}\right) &= \iint\limits_{\text{shaded region}} 6(1-y_2)\,dy_1\,dy_2 \\
&= \int_{y_1=0}^{3/4} \int_{y_2=y_1}^{1} 6(1-y_2)\,dy_2\,dy_1 \\
&= \int_0^{3/4} \left(6y_2 - 3y_2^2 \Big|_{y_1}^{1} \right) dy_1
\end{aligned}
$$

Solution: (continued)

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&= \int_0^{3/4}\left(6y_2 - 3y_2^2\Big|_{y_1}^{1}\right)dy_1 \\
&= \int_0^{3/4}\left[(6-3)-(6y_1-3y_1^2)\right]dy_1
\end{aligned}
$$

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$$= \int_0^{3/4} 3(1-y_1)^2\,dy_1 = 3 \cdot \left(-\frac{1}{3}\right)(1-y_1)^3 \Big|_0^{3/4}$$

Solution: (continued)

$$= -(1 - y_1)^3 \Big|_0^{3/4}$$

Solution: (continued)

$$= -(1 - y_1)^3\Big|_0^{3/4} = \left[-\left(1 - \frac{3}{4}\right)^3\right] - \left[-(1-0)^3\right]$$

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$$= -\left(\frac{1}{4}\right)^3 + 1 = -\frac{1}{64} + 1$$

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Next: $P\left(Y_2 \le 1/2, Y_1 \le 3/4\right)$.

Solution: (continued)

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Again, draw the region:

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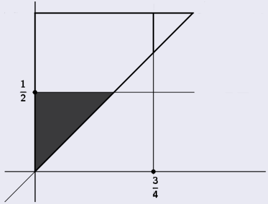

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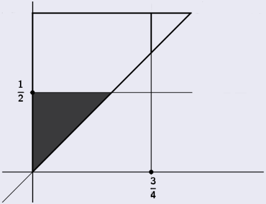
Again, draw the region:
Notice for this region, the cutoff $Y_1 \le 3/4$ doesn't matter!

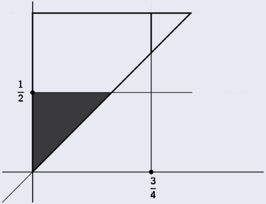


Solution: (continued)

$$= -(1-y_1)^3\Big|_0^{3/4} = \left[-\left(1-\frac{3}{4}\right)^3\right] - \left[-(1-0)^3\right]$$

$$= -\left(\frac{1}{4}\right)^3 + 1 = -\frac{1}{64} + 1 = \frac{63}{64}.$$

Next: $P\left(Y_2 \le 1/2, Y_1 \le 3/4\right)$.
Again, draw the region:
Notice for this region, the cutoff $Y_1 \le 3/4$ doesn't matter!



Remark: We might have missed this if we hadn't drawn the region: this is a common mistake.

Solution: (continued)

$$P\left(Y_2 \le \frac{1}{2}, Y_1 \le \frac{3}{4}\right) = \iint\limits_{\text{shaded region}} 6(1-y_2)\,dy_1\,dy_2$$

Solution: (continued)

$$P\left(Y_2 \le \frac{1}{2}, Y_1 \le \frac{3}{4}\right) = \iint\limits_{\text{shaded region}} 6(1-y_2)\,dy_1\,dy_2$$

$$= \int_{y_1=0}^{1/2} \int_{y_2=y_1}^{1/2} 6(1-y_2)\,dy_2\,dy_1$$

Solution: (continued)

$$P\left(Y_2 \le \frac{1}{2}, Y_1 \le \frac{3}{4}\right) = \iint\limits_{\text{shaded region}} 6(1-y_2)\,dy_1\,dy_2$$

$$= \int_{y_1=0}^{1/2} \int_{y_2=y_1}^{1/2} 6(1-y_2)\,dy_2\,dy_1$$

$$= \int_0^{1/2} \left(6y_2 - 3y_2^2 \Big|_{y_1}^{1/2} \right) dy_1$$

Solution: (continued)

$$P\left(Y_2 \le \frac{1}{2}, Y_1 \le \frac{3}{4}\right) = \iint_{\text{shaded region}} 6(1-y_2)\,dy_1\,dy_2$$

$$= \int_{y_1=0}^{1/2} \int_{y_2=y_1}^{1/2} 6(1-y_2)\,dy_2\,dy_1$$

$$= \int_0^{1/2} \left(6y_2 - 3y_2^2\Big|_{y_1}^{1/2}\right) dy_1$$

$$= \int_0^{1/2} \left[\left(6 \cdot \frac{1}{2} - 3 \cdot \left(\frac{1}{2}\right)^2\right) - (6y_1 - 3y_1^2)\right] dy_1$$

Solution: (continued)

$$P\left(Y_2 \le \frac{1}{2}, Y_1 \le \frac{3}{4}\right) = \iint_{\text{shaded region}} 6(1-y_2)\,dy_1\,dy_2$$

$$= \int_{y_1=0}^{1/2} \int_{y_2=y_1}^{1/2} 6(1-y_2)\,dy_2\,dy_1$$

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$$= \int_0^{1/2} \left[\left(6 \cdot \frac{1}{2} - 3 \cdot \left(\frac{1}{2}\right)^2\right) - (6y_1 - 3y_1^2)\right] dy_1$$

$$= \int_0^{1/2} \left[\left(3 - \frac{3}{4}\right) - 6y_1 + 3y_1^2\right] dy_1$$

Solution: (continued)

$$
\begin{aligned}
P\left(Y_2 \le \frac{1}{2}, Y_1 \le \frac{3}{4}\right) &= \iint\limits_{\text{shaded region}} 6(1-y_2)\,dy_1\,dy_2 \\
&= \int_{y_1=0}^{1/2}\int_{y_2=y_1}^{1/2} 6(1-y_2)\,dy_2\,dy_1 \\
&= \int_0^{1/2} \left(6y_2 - 3y_2^2\Big|_{y_1}^{1/2}\right) dy_1 \\
&= \int_0^{1/2} \left[\left(6\cdot\frac{1}{2} - 3\cdot\left(\frac{1}{2}\right)^2\right) - (6y_1 - 3y_1^2)\right] dy_1 \\
&= \int_0^{1/2} \left[\left(3 - \frac{3}{4}\right) - 6y_1 + 3y_1^2\right] dy_1 \\
&= \int_0^{1/2} \left[\left(-\frac{3}{4}\right) + (3 - 6y_1 + 3y_1^2)\right] dy_1
\end{aligned}
$$

Solution: (continued)

$$= \int_0^{1/2} \left(-\frac{3}{4}\right) dy_1 + \int_0^{1/2} 3(1 - y_1^2) dy_1$$

Solution: (continued)

$$= \int_0^{1/2} \left(-\frac{3}{4}\right) dy_1 + \int_0^{1/2} 3(1 - y_1^2) dy_1$$

$$= 3 \cdot \left(-\frac{1}{3}\right) (1 - y_1)^3 \Big|_0^{1/2} + \frac{1}{2} \cdot \left(-\frac{3}{4}\right)$$

Solution: (continued)

$$= \int_0^{1/2} \left(-\frac{3}{4}\right) dy_1 + \int_0^{1/2} 3(1 - y_1^2) dy_1$$

$$= 3 \cdot \left(-\frac{1}{3}\right) (1 - y_1)^3 \Big|_0^{1/2} + \frac{1}{2} \cdot \left(-\frac{3}{4}\right)$$

$$= -\left(1 - \frac{1}{2}\right)^3 - \left(-(1 - 0)^3\right) - \frac{3}{8}$$

Solution: (continued)

$$
\begin{aligned}
&= \int_0^{1/2} \left(-\frac{3}{4}\right) dy_1 + \int_0^{1/2} 3(1-y_1^2) dy_1 \\
&= 3 \cdot \left(-\frac{1}{3}\right) (1-y_1)^3 \Big|_0^{1/2} + \frac{1}{2} \cdot \left(-\frac{3}{4}\right) \\
&= -\left(1-\frac{1}{2}\right)^3 - \left(-(1-0)^3\right) - \frac{3}{8} \\
&= -\frac{1}{8} - (-1) - \frac{3}{8}
\end{aligned}
$$

Solution: (continued)

$$
\begin{aligned}
&= \int_0^{1/2} \left(-\frac{3}{4}\right) dy_1 + \int_0^{1/2} 3(1-y_1^2)dy_1 \\
&= 3 \cdot \left(-\frac{1}{3}\right) (1-y_1)^3 \Big|_0^{1/2} + \frac{1}{2} \cdot \left(-\frac{3}{4}\right) \\
&= -\left(1-\frac{1}{2}\right)^3 - \left(-(1-0)^3\right) - \frac{3}{8} \\
&= -\frac{1}{8} - (-1) - \frac{3}{8} = \boxed{\frac{1}{2}}.
\end{aligned}
$$

Solution: (continued)

$$= \int_0^{1/2} \left(-\frac{3}{4}\right) dy_1 + \int_0^{1/2} 3(1 - y_1^2) dy_1$$

$$= 3 \cdot \left(-\frac{1}{3}\right) (1 - y_1)^3 \Big|_0^{1/2} + \frac{1}{2} \cdot \left(-\frac{3}{4}\right)$$

$$= -\left(1 - \frac{1}{2}\right)^3 - \left(-(1 - 0)^3\right) - \frac{3}{8}$$

$$= -\frac{1}{8} - (-1) - \frac{3}{8} = \boxed{\frac{1}{2}}.$$

To finish, divide this by the previous fraction.

Solution: (continued)

$$
\begin{aligned}
&= \int_0^{1/2} \left(-\frac{3}{4}\right) dy_1 + \int_0^{1/2} 3(1-y_1^2) dy_1 \\
&= 3 \cdot \left(-\frac{1}{3}\right) (1-y_1)^3 \Big|_0^{1/2} + \frac{1}{2} \cdot \left(-\frac{3}{4}\right) \\
&= -\left(1-\frac{1}{2}\right)^3 - \left(-(1-0)^3\right) - \frac{3}{8} \\
&= -\frac{1}{8} - (-1) - \frac{3}{8} = \boxed{\frac{1}{2}}.
\end{aligned}
$$

To finish, divide this by the previous fraction.

Final answer: $\dfrac{32}{63}$.

Practice Problem 2:

Given that the joint density function of Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} \frac{y_1}{8} e^{-\frac{(y_1+y_2)}{2}} & y_1 > 0, y_2 > 0, \\ 0 & \text{elsewhere,} \end{cases}$$

find $E\left[\frac{Y_2}{Y_1}\right]$.

Practice Problem 2:

Given that the joint density function of Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} \frac{y_1}{8} e^{-\frac{(y_1+y_2)}{2}} & y_1 > 0, y_2 > 0, \\ 0 & \text{elsewhere,} \end{cases}$$

find $E\left[\frac{Y_2}{Y_1}\right]$. [Hint: Y_1, Y_2 are independent.]

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Solution:

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Given that the joint density function of Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} \dfrac{y_1}{8} e^{-\frac{(y_1+y_2)}{2}} & y_1 > 0, y_2 > 0, \\ 0 & \text{elsewhere,} \end{cases}$$

find $E\left[\frac{Y_2}{Y_1}\right]$. [Hint: Y_1, Y_2 are independent.]

Solution:

Since Y_1, Y_2 are independent, so are Y_2, $\frac{1}{Y_1}$.

Practice Problem 2:

Given that the joint density function of Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} \frac{y_1}{8} e^{-\frac{(y_1+y_2)}{2}} & y_1 > 0, y_2 > 0, \\ 0 & \text{elsewhere,} \end{cases}$$

find $E\left[\frac{Y_2}{Y_1}\right]$. [Hint: Y_1, Y_2 are independent.]

Solution:

Since Y_1, Y_2 are independent, so are $Y_2, \frac{1}{Y_1}$. So $E\left[\frac{Y_2}{Y_1}\right] = E[Y_2] \cdot E\left[\frac{1}{Y_1}\right]$.

Practice Problem 2:

Given that the joint density function of Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} \frac{y_1}{8} e^{-\frac{(y_1+y_2)}{2}} & y_1 > 0, y_2 > 0, \\ 0 & \text{elsewhere,} \end{cases}$$

find $E\left[\frac{Y_2}{Y_1}\right]$. [Hint: Y_1, Y_2 are independent.]

Solution:

Since Y_1, Y_2 are independent, so are $Y_2, \frac{1}{Y_1}$. So $E\left[\frac{Y_2}{Y_1}\right] = E[Y_2] \cdot E\left[\frac{1}{Y_1}\right]$.

To find $E[Y_2]$, we first find the marginal density function

$$f_2(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1.$$

Practice Problem 2:

Given that the joint density function of Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} \dfrac{y_1}{8} e^{-\frac{(y_1+y_2)}{2}} & y_1 > 0, y_2 > 0, \\ 0 & \text{elsewhere,} \end{cases}$$

find $E\left[\frac{Y_2}{Y_1}\right]$. [Hint: Y_1, Y_2 are independent.]

Solution:

Since Y_1, Y_2 are independent, so are $Y_2, \frac{1}{Y_1}$. So $E\left[\frac{Y_2}{Y_1}\right] = E[Y_2] \cdot E\left[\frac{1}{Y_1}\right]$.

To find $E[Y_2]$, we first find the marginal density function

$$f_2(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1.$$

Then

$$E[Y_2] = \int_{-\infty}^{\infty} y_2 f_2(y_2)\, dy_2.$$

Solution: (continued)

We compute

$$f_2(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1$$

Solution: (continued)

We compute

$$\begin{aligned} f_2(y_2) &= \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1 \\ &= \int_0^{\infty} \frac{y_1}{8} e^{-\frac{(y_1+y_2)}{2}}\, dy_1 \end{aligned}$$

Solution: (continued)

We compute

$$
\begin{aligned}
f_2(y_2) &= \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1 \\
&= \int_0^{\infty} \frac{y_1}{8} e^{-\frac{(y_1+y_2)}{2}}\, dy_1 = \frac{1}{8} e^{-\frac{y_2}{2}} \int_0^{\infty} y_1 e^{-\frac{y_1}{2}}\, dy_1
\end{aligned}
$$

Solution: (continued)

We compute

$$f_2(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1$$

$$= \int_0^{\infty} \frac{y_1}{8} e^{-\frac{(y_1+y_2)}{2}}\, dy_1 = \frac{1}{8} e^{-\frac{y_2}{2}} \int_0^{\infty} y_1 e^{-\frac{y_1}{2}}\, dy_1$$

$$v = \tfrac{y_1}{2} \qquad 2v = y$$

Solution: (continued)

We compute

$$f_2(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1$$

$$= \int_0^{\infty} \frac{y_1}{8} e^{-\frac{(y_1+y_2)}{2}}\, dy_1 = \frac{1}{8} e^{-\frac{y_2}{2}} \int_0^{\infty} y_1 e^{-\frac{y_1}{2}}\, dy_1$$

$$\begin{aligned} v &= \tfrac{y_1}{2} & 2v &= y \\ dv &= \tfrac{1}{2} dy_1 & 2dv &= dy_1 \end{aligned}$$

Solution: (continued)

We compute

$$\begin{aligned} f_2(y_2) &= \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1 \\ &= \int_0^{\infty} \frac{y_1}{8} e^{-\frac{(y_1+y_2)}{2}}\, dy_1 = \frac{1}{8} e^{-\frac{y_2}{2}} \int_0^{\infty} y_1 e^{-\frac{y_1}{2}}\, dy_1 \\ &= \frac{1}{8} e^{-\frac{y_2}{2}} \int_{v=0}^{\infty} 2v e^{-v}\, 2dv \qquad \begin{array}{rl} v = \frac{y_1}{2} & 2v = y \\ dv = \frac{1}{2} dy_1 & 2dv = dy_1 \end{array} \end{aligned}$$

Solution: (continued)

We compute

$$
\begin{aligned}
f_2(y_2) &= \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1 \\
&= \int_0^{\infty} \frac{y_1}{8} e^{-\frac{(y_1+y_2)}{2}}\, dy_1 = \frac{1}{8} e^{-\frac{y_2}{2}} \int_0^{\infty} y_1 e^{-\frac{y_1}{2}}\, dy_1 \\
&= \frac{1}{8} e^{-\frac{y_2}{2}} \int_{v=0}^{\infty} 2v e^{-v}\, 2dv \qquad \begin{matrix} v = \frac{y_1}{2} & 2v = y \\ dv = \frac{1}{2} dy_1 & 2dv = dy_1 \end{matrix} \\
&= \frac{1}{2} e^{-\frac{y_2}{2}} \int_{v=0}^{\infty} v e^{-v}\, dv
\end{aligned}
$$

Solution: (continued)

We compute

$$\begin{aligned} f_2(y_2) &= \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1 \\ &= \int_0^{\infty} \frac{y_1}{8} e^{-\frac{(y_1+y_2)}{2}}\, dy_1 = \frac{1}{8} e^{-\frac{y_2}{2}} \int_0^{\infty} y_1 e^{-\frac{y_1}{2}}\, dy_1 \\ &= \frac{1}{8} e^{-\frac{y_2}{2}} \int_{v=0}^{\infty} 2v e^{-v}\, 2dv \qquad \begin{aligned} v &= \tfrac{y_1}{2} \\ dv &= \tfrac{1}{2} dy_1 \end{aligned} \quad \begin{aligned} 2v &= y \\ 2dv &= dy_1 \end{aligned} \\ &= \frac{1}{2} e^{-\frac{y_2}{2}} \int_{v=0}^{\infty} v e^{-v}\, dv = \frac{1}{2} e^{-y_2/2} \left[-v e^{-v} \Big|_0^{\infty} - \int_0^{\infty} e^{-v}\, dv \right] \end{aligned}$$

Solution: (continued)

We compute

$$
\begin{aligned}
f_2(y_2) &= \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1 \\
&= \int_0^{\infty} \frac{y_1}{8} e^{-\frac{(y_1+y_2)}{2}}\, dy_1 = \frac{1}{8} e^{-\frac{y_2}{2}} \int_0^{\infty} y_1 e^{-\frac{y_1}{2}}\, dy_1 \\
&= \frac{1}{8} e^{-\frac{y_2}{2}} \int_{v=0}^{\infty} 2v e^{-v}\, 2dv \qquad \begin{array}{rl} v = \frac{y_1}{2} & \quad 2v = y \\ dv = \frac{1}{2} dy_1 & \quad 2dv = dy_1 \end{array} \\
&= \frac{1}{2} e^{-\frac{y_2}{2}} \int_{v=0}^{\infty} v e^{-v}\, dv = \frac{1}{2} e^{-y_2/2} \left[-v e^{-v} \Big|_0^{\infty} - \int_0^{\infty} e^{-v}\, dv \right] \\
&= \frac{1}{2} e^{-\frac{y_2}{2}}.
\end{aligned}
$$

Solution: (continued)

We compute

$$
\begin{aligned}
f_2(y_2) &= \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1 \\
&= \int_0^{\infty} \frac{y_1}{8} e^{-\frac{(y_1+y_2)}{2}}\, dy_1 = \frac{1}{8} e^{-\frac{y_2}{2}} \int_0^{\infty} y_1 e^{-\frac{y_1}{2}}\, dy_1 \\
&= \frac{1}{8} e^{-\frac{y_2}{2}} \int_{v=0}^{\infty} 2ve^{-v}\, 2dv \qquad \begin{array}{rl} v = & \frac{y_1}{2} \\ dv = & \frac{1}{2} dy_1 \end{array} \quad \begin{array}{rl} 2v = & y \\ 2dv = & dy_1 \end{array} \\
&= \frac{1}{2} e^{-\frac{y_2}{2}} \int_{v=0}^{\infty} ve^{-v}\, dv = \frac{1}{2} e^{-y_2/2} \left[-ve^{-v} \Big|_0^{\infty} - \int_0^{\infty} e^{-v}\, dv \right] \\
&= \frac{1}{2} e^{-\frac{y_2}{2}}. \qquad \text{(Note } f_2(y_2) = 0 \text{ for } y_2 < 0.)
\end{aligned}
$$

Solution: (continued)

So

$$E[Y_2] = \int_{-\infty}^{\infty} y_2 f_2(y_2)\, dy_2$$

Solution: (continued)

So

$$E[Y_2] = \int_{-\infty}^{\infty} y_2 f_2(y_2)\, dy_2 = \int_0^{\infty} \frac{1}{2} y_2 e^{-\frac{y_2}{2}}\, dy_2$$

Solution: (continued)

So

$$E[Y_2] = \int_{-\infty}^{\infty} y_2 f_2(y_2)\, dy_2 = \int_0^{\infty} \frac{1}{2} y_2 e^{-\frac{y_2}{2}}\, dy_2$$

$$= \frac{1}{2} \int_0^{\infty} y_2 e^{-\frac{y_2}{2}}\, dy_2$$

Solution: (continued)

So

$$E[Y_2] = \int_{-\infty}^{\infty} y_2 f_2(y_2)\, dy_2 = \int_0^{\infty} \frac{1}{2} y_2 e^{-\frac{y_2}{2}}\, dy_2$$

$$= \frac{1}{2} \underbrace{\int_0^{\infty} y_2 e^{-\frac{y_2}{2}}\, dy_2} \quad \longleftarrow$$

This is exactly the integral we just did with y_1 replaced by y_2.

Solution: (continued)

So

$$
\begin{aligned}
E[Y_2] &= \int_{-\infty}^{\infty} y_2 f_2(y_2)\, dy_2 = \int_0^{\infty} \frac{1}{2} y_2 e^{-\frac{y_2}{2}}\, dy_2 \\
&= \frac{1}{2} \underbrace{\int_0^{\infty} y_2 e^{-\frac{y_2}{2}}\, dy_2} \qquad \longleftarrow \\
&= \frac{1}{2} \cdot 4
\end{aligned}
$$

This is exactly the integral we just did with y_1 replaced by y_2.

Solution: (continued)

So

$$E[Y_2] = \int_{-\infty}^{\infty} y_2 f_2(y_2)\, dy_2 = \int_0^{\infty} \frac{1}{2} y_2 e^{-\frac{y_2}{2}}\, dy_2$$

$$= \frac{1}{2} \underbrace{\int_0^{\infty} y_2 e^{-\frac{y_2}{2}}\, dy_2} \longleftarrow$$

This is exactly the integral we just did with y_1 replaced by y_2.

$$= \frac{1}{2} \cdot 4 = \boxed{2}.$$

Solution: (continued)

So

$$E[Y_2] = \int_{-\infty}^{\infty} y_2 f_2(y_2)\, dy_2 = \int_0^{\infty} \frac{1}{2} y_2 e^{-\frac{y_2}{2}}\, dy_2$$

$$= \frac{1}{2} \underbrace{\int_0^{\infty} y_2 e^{-\frac{y_2}{2}}\, dy_2} \longleftarrow$$

This is exactly the integral we just did with y_1 replaced by y_2.

$$= \frac{1}{2} \cdot 4 = \boxed{2}.$$

Similarly, compute $E\left[\frac{1}{Y_1}\right]$.

Solution: (continued)

So

$$E[Y_2] = \int_{-\infty}^{\infty} y_2 f_2(y_2)\, dy_2 = \int_0^{\infty} \frac{1}{2} y_2 e^{-\frac{y_2}{2}}\, dy_2$$

$$= \frac{1}{2} \underbrace{\int_0^{\infty} y_2 e^{-\frac{y_2}{2}}\, dy_2} \quad \longleftarrow \quad \text{This is exactly the integral we just did with } y_1 \text{ replaced by } y_2.$$

$$= \frac{1}{2} \cdot 4 = \boxed{2}.$$

Similarly, compute $E\left[\frac{1}{Y_1}\right]$.

Now multiply these to get the final answer:

Solution: (continued)

So

$$E[Y_2] = \int_{-\infty}^{\infty} y_2 f_2(y_2)\,dy_2 = \int_0^{\infty} \frac{1}{2} y_2 e^{-\frac{y_2}{2}}\,dy_2$$

$$= \frac{1}{2} \underbrace{\int_0^{\infty} y_2 e^{-\frac{y_2}{2}}\,dy_2} \qquad \longleftarrow$$

This is exactly the integral we just did with y_1 replaced by y_2.

$$= \frac{1}{2} \cdot 4 = \boxed{2}.$$

Similarly, compute $E\left[\frac{1}{Y_1}\right]$. $\left(= \frac{1}{2}\right)$.

Now multiply these to get the final answer:

Solution: (continued)

So

$$E[Y_2] = \int_{-\infty}^{\infty} y_2 f_2(y_2)\, dy_2 = \int_0^{\infty} \frac{1}{2} y_2 e^{-\frac{y_2}{2}}\, dy_2$$

$$= \frac{1}{2} \underbrace{\int_0^{\infty} y_2 e^{-\frac{y_2}{2}}\, dy_2}$$

$\longleftarrow$ This is exactly the integral we just did with y_1 replaced by y_2.

$$= \frac{1}{2} \cdot 4 = \boxed{2}.$$

Similarly, compute $E\left[\frac{1}{Y_1}\right]$. $\left(= \frac{1}{2}\right)$.

Now multiply these to get the final answer: $E\left[\frac{Y_2}{Y_1}\right] = 1$.

Exercise 5.27: (Relevant for MATH 448)

Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables with $E[Y_i] = \mu$ and $V[Y_i] = \sigma^2$. Consider the new RV $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$. What are the mean and the variance of $\overline{Y}$?

Exercise 5.27: (Relevant for MATH 448)

Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables with $E[Y_i] = \mu$ and $V[Y_i] = \sigma^2$. Consider the new RV $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$. What are the mean and the variance of $\overline{Y}$?

Solution:

Exercise 5.27: (Relevant for MATH 448)

Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables with $E[Y_i] = \mu$ and $V[Y_i] = \sigma^2$. Consider the new RV $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$. What are the mean and the variance of $\overline{Y}$?

Solution:

$$E\left[\overline{Y}\right] = E\left[\frac{1}{n}\sum_{i=1}^{n} Y_i\right]$$

Exercise 5.27: (Relevant for MATH 448)

Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables with $E[Y_i] = \mu$ and $V[Y_i] = \sigma^2$. Consider the new RV $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$. What are the mean and the variance of $\overline{Y}$?

Solution:

$$E\left[\overline{Y}\right] = E\left[\frac{1}{n}\sum_{i=1}^{n} Y_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[Y_i]$$

Exercise 5.27: (Relevant for MATH 448)

Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables with $E[Y_i] = \mu$ and $V[Y_i] = \sigma^2$. Consider the new RV $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$. What are the mean and the variance of $\overline{Y}$?

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$$E\left[\overline{Y}\right] = E\left[\frac{1}{n}\sum_{i=1}^{n} Y_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[Y_i] = \frac{1}{n}\sum_{i=1}^{n}\mu$$

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Note that in the correct version of this computation, we used $V[aY] = a^2 V[Y]$, and that if X, Y are independent, then X and $-Y$ are independent.

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Suppose that an urn contains r red balls and $N - r$ black balls. A random sample of n balls is drawn without replacement and Y, the number of red balls in the sample, is observed. Find the mean and variance of Y.

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We have learnt that for a hypergeometric distribution Y,

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Let's consider the dependence more carefully:

$$P(X_2 = 1, X_1 = 1) = P(X_2 = 1 \mid X_1 = 1) \cdot P(X_1 = 1)$$

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We can now start thinking about

$$V[Y] = \mathrm{Cov}(Y, Y)$$

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Since $X_i = 0$ or 1, $E[X_iX_j] = \dfrac{(r-1)r}{(N-1)N}$.

We can now start thinking about

$$V[Y] = \mathrm{Cov}(Y, Y) = \mathrm{Cov}(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i)$$

Solution: (continued)

$$E[Y] = E[X_1 + \cdots + X_n] = E[X_1] + \cdots + E[X_n] = \frac{nr}{N}.$$

Note that the X_i are dependent on one another and we can use linearity of E anyway.

Let's consider the dependence more carefully:

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$$V[Y] = \mathrm{Cov}(Y, Y) = \mathrm{Cov}(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \sum_{j=1}^{n} \mathrm{Cov}(X_i, X_j).$$

Solution: (continued)

If $i \neq j$, then

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$$V[Y] = n \cdot \left[\frac{r}{N} - \left(\frac{r}{N}\right)^2\right] + n(n-1) \cdot \left[\frac{(r-1)r}{(N-1)N} - \left(\frac{r}{N}\right)^2\right]$$

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$$= \frac{nr}{N} \cdot \left(\left[1 - \frac{r}{N}\right] + (n-1)\left[\frac{r-1}{N-1} - \frac{r}{N}\right]\right)$$

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Solution: (continued)

$$= \frac{nr}{N} \cdot \left(\frac{N - r - (n-1)r}{N} + \frac{(n-1)(r-1)}{N-1} \right)$$

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\begin{aligned}
&= \frac{nr}{N} \cdot \left(\frac{N - r - (n-1)r}{N} + \frac{(n-1)(r-1)}{N-1} \right) \\
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&= \frac{nr}{N} \cdot \left(\frac{(N - nr)(N-1) + N(n-1)(r-1)}{N(N-1)} \right) \\
&= \frac{nr}{N} \cdot \left(\frac{(N^2 - Nnr - N + nr) + (Nnr - Nn - Nr + N)}{N(N-1)} \right)
\end{aligned}
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Solution: (continued)

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&= \frac{nr}{N} \cdot \left(\frac{(N - nr)(N-1) + N(n-1)(r-1)}{N(N-1)} \right) \\
&= \frac{nr}{N} \cdot \left(\frac{(N^2 - \cancel{Nnr} - \cancel{N} + nr) + (\cancel{Nnr} - Nn - Nr + \cancel{N})}{N(N-1)} \right) \\
&= \frac{nr}{N} \left[\frac{(N-r)}{N} \frac{(N-n)}{(N-1)} \right], \qquad \text{as claimed.}
\end{aligned}
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Assume that $p_1, \ldots, p_k$ are such that $\sum_{i=1}^{n} p_i = 1$, and $p_i > 0$ for $i = 1, \ldots, k$. The random variables $Y_1, \ldots, Y_k$ are said to have a multinomial distribution with parameters n and $p_1, \ldots, p_k$ if the joint probability function of $Y_1, \ldots, Y_k$ is given by

$$p(y_1, \ldots, y_k) = \frac{n!}{y_1! \ldots y_k!} p_1^{y_1} \ldots p_k^{y_k},$$

where, for each i, $y_i = 0, 1, \ldots, n$ and $\sum_{i=1}^{k} y_i = n$.

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By thinking of outcome type i as success, and anything else as failure, we see that the marginal distribution of each Y_i is binomial with parameters n (the number of trials) and p_i (the probability of outcome type i).

Theorem (5.13)

If $Y_1, \ldots, Y_k$ have a multinomial distribution with parameters n and $p_1, \ldots, p_k$, then

(1) $E[Y_i] = np_i, V[Y_i] = np_i q_i$, *where* $q_i = 1 - p_i$.

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Proof: (Part (2) of Theorem 5.13)

Define

$$U_i = \begin{cases} 1 & \text{if trial } i \text{ results in outcome } s, \\ 0 & \text{otherwise} \end{cases}, \quad V_j = \begin{cases} 1 & \text{if trial } j \text{ results in outcome } t, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: (continued)

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Now $\mathrm{Cov}(U_i, V_i) = E[U_i V_i] - E[U_i]E[V_i]$.

Proof: (continued)

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Now $\mathrm{Cov}(U_i, V_i) = E[U_i V_i] - E[U_i]E[V_i]$. Since U_i and V_i cannot both be 1, $E[U_i V_i] = 0$.

Proof: (continued)

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Exercise 5.119

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(d) To check for the rat's preference between exits 2 and 3, we may look at $Y_2 - Y_3$. Find $E[Y_2 - Y_3]$ and $V[Y_2 - Y_3]$ for general n.

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If you think of X and Y as being like vectors, you can think of $\rho_{X,Y}$ as being like the cosine of the angle between them.

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Suppose $\rho_{X,Y} = 0.9$ and $\rho_{Y,Z} = 0.8$. What is the minimum possible value of $\rho_{X,Z}$?

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Solution:

Suppose that we write $\theta_{X,Y}$ and $\theta_{Y,Z}$ for the angles between the RVs X, Y and Y, Z.

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Suppose that we write $\theta_{X,Y}$ and $\theta_{Y,Z}$ for the angles between the RVs X, Y and Y, Z. Then $\theta_{X,Y} = \cos^{-1}(0.9)$ and $\theta_{Y,Z} = \cos^{-1}(0.8)$.

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The Bivariate Normal Distribution

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Definition (Bivariate Normal Distribution)

Two continuous RVs Y_1, Y_2 are said to have the bivariate normal distribution if the density function is given by

$$f(y_1, y_2) = \frac{e^{-Q/2}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}, \quad -\infty < y_1 < \infty, -\infty < y_2 < \infty,$$

where

$$Q = \frac{1}{1-\rho^2}\left[\frac{(y_1-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(y_1-\mu_1)(y_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(y_2-\mu_2)^2}{\sigma_2^2}\right].$$

Thus the bivariate normal distribution is a function of five parameters: $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2$, and ρ.

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- With a bit of somewhat tedious integration, we can also show that
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- This distribution is special, in the sense that, if Y_1 and Y_2 have a bivariate normal distribution, they are independent if and only if their covariance (equivalently, $\rho = \rho_{Y_1, Y_2}$) is zero. Zero covariance does not imply independence in general.

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Check that $k = 2$ gives the bivariate normal distribution we have just seen.

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Assuming that everything is defined and we haven't divided by zero, we could compute the expectation of $E[Y_1 \mid Y_2 = y_2]$, because it is a function of Y_2.

Theorem (5.14)

Let Y_1 and Y_2 denote random variables. Then

$$E[Y_1] = E[E[Y_1 \mid Y_2]],$$

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where on the right-hand side the inside expectation is with respect to the conditional distribution of Y_1 given Y_2 and the outside expectation is with respect to the distribution of Y_2.

Proof: (continuous case; the discrete case is analogous.)

$$\begin{aligned}
E[E[Y_1 \mid Y_2]] &= \int_{-\infty}^{\infty} E[Y_1 \mid Y_2] f_2(y_2)\, dy_2 \\
&= \int_{y_2=-\infty}^{\infty} \left[\int_{y_1=-\infty}^{\infty} y_1 f(y_1 \mid y_2)\, dy_1 \right] f_2(y_2)\, dy_2 \\
&= \int_{y_2=-\infty}^{\infty} \int_{y_1=-\infty}^{\infty} y_1 \frac{f(y_1, y_2)}{\cancel{f_2(y_2)}} \cancel{f_2(y_2)}\, dy_1\, dy_2 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y_1 f(y_1, y_2)\, dy_1\, dy_2 = E[Y_1].
\end{aligned}$$

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The formula in Theorem 5.14 was a relationship between the unconditional expectation $E[Y_1]$ and the conditional expectation $E[Y_1 \mid Y_2]$. There is a more complicated relation between the unconditional variance $V[Y]$ and the conditional variance $V[Y_1 \mid Y_2]$:

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The formula in Theorem 5.14 was a relationship between the unconditional expectation $E[Y_1]$ and the conditional expectation $E[Y_1 \mid Y_2]$. There is a more complicated relation between the unconditional variance $V[Y]$ and the conditional variance $V[Y_1 \mid Y_2]$:

Theorem (5.15)

$$V[Y_1] = E[V[Y_1 \mid Y_2]] + V[E[Y_1 \mid Y_2]].$$

Remark:

There are similar, but more complicated relationships between conditional and unconditional higher moments $E[Y_1^3]$, etc.

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Recall

$$V[Y_1 \mid Y_2] = E[Y_1^2 \mid Y_2] - E[Y_1 \mid Y_2]^2.$$

Then

$$E[V[Y_1 \mid Y_2]] = E[E[Y_1^2 \mid Y_2]] - E[E[Y_1 \mid Y_2]^2].$$

By definition,

$$V[E[Y_1 \mid Y_2]] = E[E[Y_1^2 \mid Y_2]^2] - E[E[Y_1 \mid Y_2]]^2.$$

Proof: (continued)

The variance of Y_1 is

$$V[Y_1] = E[Y_1^2] - E[Y_1]^2$$

Proof: (continued)

The variance of Y_1 is

$$\begin{aligned} V[Y_1] &= E[Y_1^2] - E[Y_1]^2 \\ &= E[E[Y_1^2 \mid Y_2]] - E[E[Y_1 \mid Y_2]]^2 \end{aligned}$$

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V[Y_1] &= E[Y_1^2] - E[Y_1]^2 \\
&= E[E[Y_1^2 \mid Y_2]] - E[E[Y_1 \mid Y_2]]^2 \qquad \text{(By Theorem 5.14)} \\
&= E[E[Y_1^2 \mid Y_2]] - E[E[Y_1^2 \mid Y_2]^2] + E[E[Y_1^2 \mid Y_2]^2] \\
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\end{aligned}$$

Proof: (continued)

The variance of Y_1 is

$$\begin{aligned} V[Y_1] &= E[Y_1^2] - E[Y_1]^2 \\ &= E[E[Y_1^2 \mid Y_2]] - E[E[Y_1 \mid Y_2]]^2 \qquad \text{(By Theorem 5.14)} \\ &= \underbrace{E[E[Y_1^2 \mid Y_2]] - E[E[Y_1^2 \mid Y_2]^2]} + E[E[Y_1^2 \mid Y_2]^2] \\ &\qquad - E[E[Y_1 \mid Y_2]]^2 \\ &= \overbrace{E[V[Y_1 \mid Y_2]]} \end{aligned}$$

- By the definition of "conditional variance".

Proof: (continued)

The variance of Y_1 is

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- By the definition of "conditional variance".
- Because $E[Y_1 \mid Y_2]$ is a RV and $V[X] = E[X^2] - E[X]^2$.

Proof: (continued)

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&= \overbrace{E[V[Y_1 \mid Y_2]]} + \overbrace{V[E[Y_1 \mid Y_2]]}, \qquad \text{as claimed.}
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- By the definition of "conditional variance".
- Because $E[Y_1 \mid Y_2]$ is a RV and $V[X] = E[X^2] - E[X]^2$.

Note: Make sure to remember this result: it will help with Exercises 5.136 and 5.138.

Exercise 5.136

The number of defects per yard in a certain fabric, Y, has a Poisson distribution with parameter λ, which is assumed to be a random variable with a density function given by

$$f(\lambda) = \begin{cases} e^{-\lambda} & \lambda \geq 0, \\ 0 & \text{elsewhere.} \end{cases}$$

Find (a) the expectation, and (b) the variance of Y. (c) Is it likely that $Y \geq 9$?

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Find (a) the expectation, and (b) the variance of Y. (c) Is it likely that $Y \geq 9$?

Exercise 5.138

Assume that Y denotes the number of bacteria per cubic centimeter in a particular liquid and that Y has a Poisson distribution with parameter λ. Further assume that λ varies from location to location and has a Gamma distribution with parameters α and β, where α is a positive integer. If we randomly select a location, what is the

(a) expected number of bacteria per cubic centimeter?

(b) standard deviation of the number of bacteria per cubic centimeter?

Hints for Exercises 5.136 and 5.138:

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- Use Theorem 5.14 for parts (a) and Theorem 5.15 for parts (b).

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Exercise 5.167

Let Y_1 and Y_2 be jointly distributed random variables with finite variances.

(a) Show that

$$E[Y_1 Y_2]^2 \leq E[Y_1^2]E[Y_2^2]$$

by observing that

$$E[(tY_1 - Y_2)^2] \geq 0$$

for any real number t.

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(b) Hence prove that

$$-1 \leq \rho_{Y_1,Y_2} \leq 1$$

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Solution: (Exercise 5.167(a))

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If Y_1, Y_2 are RVs, then note that $E[(tY_1 - Y_2)^2] \geq 0$.

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$$\therefore \quad t^2 E[Y_1^2] - 2tE[Y_1 Y_2] + E[Y_2^2] \geq 0.$$

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$$b = -2E[Y_1 Y_2], \qquad a = E[Y_1^2], \qquad c = E[Y_2^2].$$

So

$$(-2E[Y_1 Y_2])^2 - 4E[Y_1^2]E[Y_2^2] \leq 0.$$

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$$\therefore \quad 4(E[Y_1 Y_2])^2 - E[Y_1^2]E[Y_2^2]) \leq 0.$$

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$$\therefore \quad E[Y_1 Y_2]^2 \leq E[Y_1^2]E[Y_2^2].$$

Solution: (Exercise 5.167(b))

Now recall, for RVs X_1, X_2,

$$\rho_{X_1,X_2} = \frac{\mathrm{Cov}(X_1, X_2)}{\sigma_{X_1}\sigma_{X_2}}$$

Solution: (Exercise 5.167(b))

Now recall, for RVs X_1, X_2,

$$\rho_{X_1,X_2} = \frac{\mathrm{Cov}(X_1, X_2)}{\sigma_{X_1}\sigma_{X_2}} = \frac{E[(X_1 - \mu_1)(X_2 - \mu_2)]}{\sqrt{E[(X_1 - \mu_1)^2]E[(X_2 - \mu_2)^2]}},$$

where $\mu_1 = E[X_1]$ and $\mu_2 = E[X_2]$.

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Now let $Y_1 = X_1 - \mu_1$ and $Y_2 = X_2 - \mu_2$.

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By Exercise 5.167(a), we know that $E[Y_1 Y_2]^2 \leq E[Y_1^2]E[Y_2^2]$,

Solution: (Exercise 5.167(b))

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$$\frac{E[Y_1 Y_2]^2}{E[Y_1^2]E[Y_2^2]} \leq 1 \implies \rho^2_{Y_1,Y_2} \leq 1,$$

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where we have used the linearity of Expectations.

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where we have used the linearity of Expectations. Thus

$$-1 \leq \rho_{Y_1,Y_2} \leq 1,$$

as desired.

Exercise 5.31:

The joint density function of Y_1 and Y_2 is given by

$$f(y_1, y_2) = \begin{cases} 30y_1y_2^2 & y_1 - 1 \leq y_2 \leq 1 - y_1, 0 \leq y_1 \leq 1, \\ 0 & \text{elsewhere.} \end{cases}$$

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(a) Show that the marginal density of Y_1 is a beta density with $\alpha = 2$ and $\beta = 4$.

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(b) Derive the marginal density of Y_2.

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(c) Derive the conditional density of Y_2 given $Y_1 = y_1$.

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(d) Find $P(Y_2 > 0 \mid Y_1 = .75)$.

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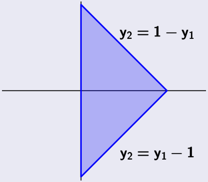
(c) Derive the conditional density of Y_2 given $Y_1 = y_1$.

(d) Find $P(Y_2 > 0 \mid Y_1 = .75)$.

Solution:

First, graph the region in which the density function is nonzero:



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All of this is part of the definition of $f_1(y_1)$!

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$$f_2(y_2) = \begin{cases} 0 & y_2 \notin [-1, 1], \\ 15 y_2^2 (1+y_2)^2 & y_2 \in [-1, 0], \\ 15 y_2^2 (1-y_2)^2 & y_2 \in [0, 1]. \end{cases}$$

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Example 5.32:

A quality control plan for an assembly line involves sampling $n = 10$ finished items per day and counting Y, the number of defectives. If p denotes the probability of observing a defective, then Y has a binomial distribution, assuming that a large number of items are produced by the line. But p varies from day to day and is assumed to have a uniform distribution on the interval from 0 to $\frac{1}{4}$. Find the expected value of Y.

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So $E[Y] = \boxed{\frac{n}{8}}$.

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So $V[Y] = E[npq] - V[np]$. Remember that $p \sim \text{Unif}\left(0, \frac{1}{4}\right)$. So

$$\begin{aligned} E[npq] = nE[pq] &= n \int_0^{1/4} y(1-y)\frac{1}{1/4}\, dy \\ &= n\int_0^{1/4} y \cdot 4dy - n\int_0^{1/4} y^2\, 4dy \end{aligned}$$

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So this can be done using the known mean and variance of a uniform RV. Instead, we directly do the above integrals:

Solution: (continued)

$$E[npq] = n \cdot 4 \cdot \left.\frac{y^2}{2}\right|_0^{1/4} - n \cdot 4 \cdot \left.\frac{y^3}{3}\right|_0^{1/4}$$

Solution: (continued)

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\begin{aligned}
E[npq] &= n \cdot 4 \cdot \frac{y^2}{2}\Big|_0^{1/4} - n \cdot 4 \cdot \frac{y^3}{3}\Big|_0^{1/4} \\
&= \frac{n}{8} - \frac{n}{48}
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So $V[Y] = \boxed{\frac{5n}{48} + \frac{n^2}{192}}$.

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Slogan for Tchebysheff's Theorem:

The probability that Y is far from its mean, where "far" is measured in units of σ, is small.

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Can you think of another RV which has mean 0 and variance 1?

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One way to "recognize" a RV is to use the moment generating function. That is, if we know (for whatever reason) that $m_X(t) = m_Y(t)$ for all t near $t = 0$, then we have that X and Y have the same distribution.

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And thus $m_2(t) = e^{(0)t + \frac{(1)^2 t^2}{2}}$.

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And thus $m_2(t) = e^{(0)t + \frac{(1)^2 t^2}{2}}$. This is the MGF of a normal RV with mean 0 and variance 1. Therefore by "uniqueness of MGF", Z is normal, with mean 0 and variance 1.

End of Chapter 5

Chapter 6

Functions of Random Variables

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Consider random variables $Y_1, \ldots, Y_n$ and a function $U(Y_1, \ldots, Y_n)$, denoted simply as U. Then three of the methods for finding the probability distribution of U are as follows:

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$$F_Y(y) = \begin{cases} y^2 & y \in [0,1], \\ 0 & y < 0, \\ 1 & y > 1. \end{cases}$$

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$$F_U(u) = \begin{cases} \frac{(u+1)^2}{9} & u \in [-1,2], \\ 0 & u < -1, \\ 1 & u > 2. \end{cases}$$

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Suppose that X and Y are independent and have the unniform distribution on the unit interval $[0, 1]$. Let $U = X + Y$. Find the density function $f_U(u)$.

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Solution:

Write $F_U(u) = P(U \leq u) = P(X + Y \leq u)$.

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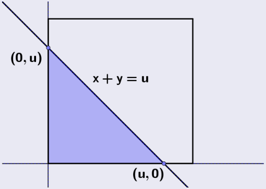

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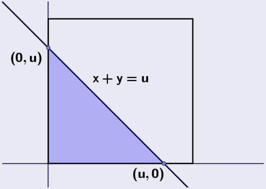

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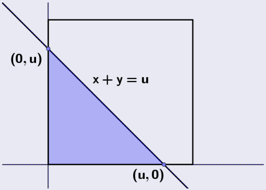

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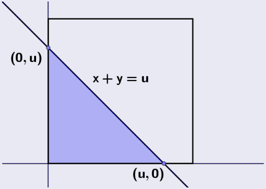

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This works for $0 \le u \le 1$.

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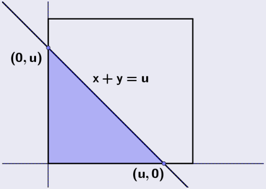

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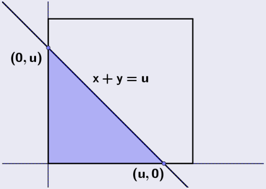

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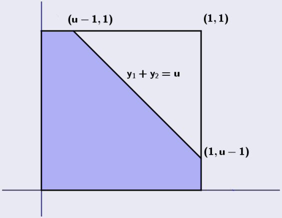

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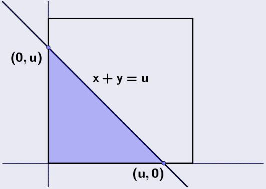

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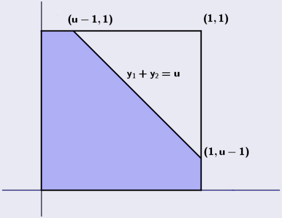

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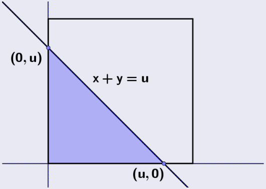

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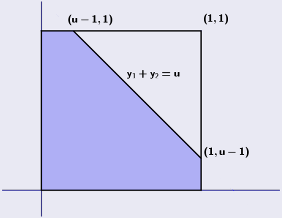

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The removed triangle has area $\dfrac{1}{2}(2-u)(2-u)$.

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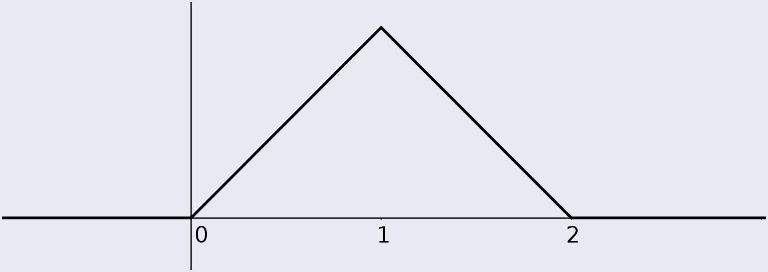

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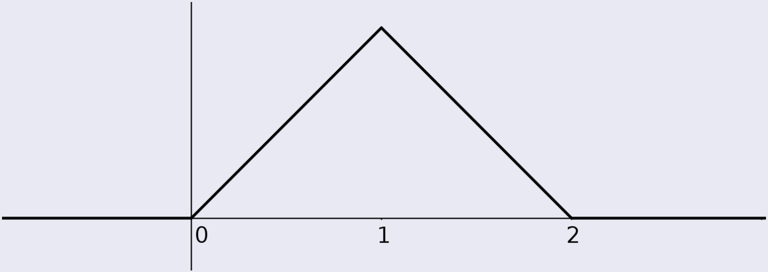

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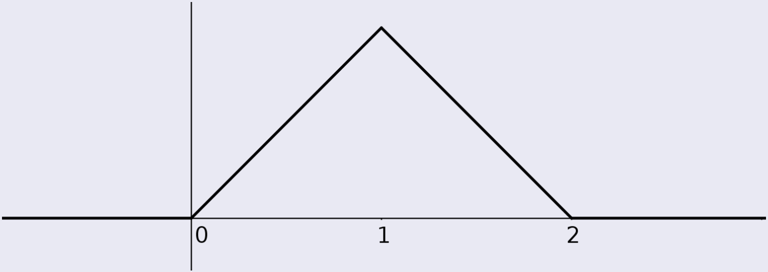

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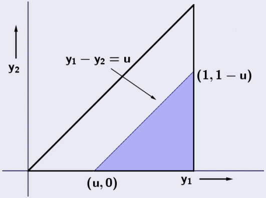

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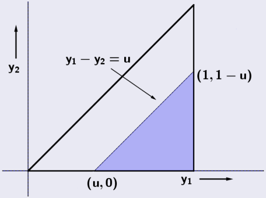

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Example

Let Y_1, Y_2 be independent exponential RVs with parameter $= 1$. Let $U = Y_1 + Y_2$. Find the PDF $f_U(u)$.

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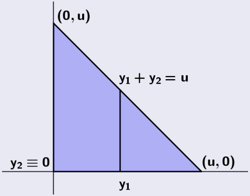

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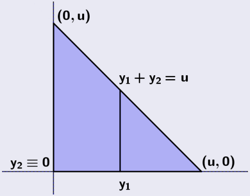

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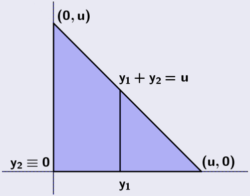

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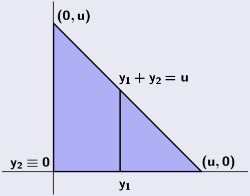

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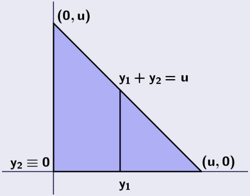

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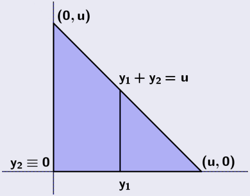

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$U = y_1 + Y_2 = h(Y_2)$.

Regard this as a function of Y_2.

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<u>Conclusion:</u> $X = \dfrac{Y-\mu}{\sigma}$ has the standard normal distribution.

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$$m_Y(t) = E\left[e^{tY}\right] = E\left[e^{tZ^2}\right]$$

$$= \int_{-\infty}^{\infty} e^{tz^2} f(z)\, dz \qquad \text{where } f \text{ is the standard normal PDF}$$

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Let Z be a standard normal RV, and let $Y = Z^2$. What is the distribution of Y?

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$$m_Y(t) = E\left[e^{tY}\right] = E\left[e^{tZ^2}\right]$$

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To integrate, use the trick: The integral of a PDF is 1.

If we can rearrange this into something that looks like (factor) $\cdot$ (normal PDF), then $\int$(factor)(PDF) $=$ (factor).

Solution: (continued)

$$\therefore \quad m_Y(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{z^2}{2}+tz^2\right)}\, dz.$$

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Thus $m_Y(t) = \sigma = \dfrac{1}{(1-2t)^{1/2}}$.

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Thus $m_Y(t) = \sigma = \dfrac{1}{(1-2t)^{1/2}}$. This is the MGF for a Gamma RV with $\alpha = \frac{1}{2}$ and $\beta = 2$. The same is a MGF of a $\chi^2[1]$ RV.

Solution: (continued)

Conclusion: The distribution of $Y = Z^2$ is $\Gamma\left(\frac{1}{2}, 2\right)$ which is the same as $\chi^2[1]$.

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In the above setting, find $E[Z^4]$.

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We have just shown that if Z is standard normal, then Z^2 is $\chi^2[1]$.

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Example

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Suppose X, Y are RVs with $X \sim \Gamma(\alpha_1, \beta)$ and $Y \sim \Gamma(\alpha_2, \beta)$, and X, Y are independent.

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Solution:

Note that

$$m_U(t) = E[e^{tU}]$$

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$$\begin{aligned} m_U(t) &= E[e^{tU}] = E[e^{t(X+Y)}] = E[e^{tX}e^{tY}] \\ &= E[e^{tX}]E[e^{tY}] \quad \text{(by independence)} \end{aligned}$$

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Suppose X, Y are RVs with $X \sim \Gamma(\alpha_1, \beta)$ and $Y \sim \Gamma(\alpha_2, \beta)$, and X, Y are independent. Let $U = X + Y$. What is the distribution of U?

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This is the MGF of a $\Gamma(\alpha_1 + \alpha_2, \beta)$ RV.

Example

Suppose X, Y are RVs with $X \sim \Gamma(\alpha_1, \beta)$ and $Y \sim \Gamma(\alpha_2, \beta)$, and X, Y are independent. Let $U = X + Y$. What is the distribution of U?

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Note that

$$\begin{aligned} m_U(t) &= E[e^{tU}] = E[e^{t(X+Y)}] = E[e^{tX} e^{tY}] \\ &= E[e^{tX}]E[e^{tY}] \quad \text{(by independence)} \quad = m_X(t)M_Y(t) \\ &= \frac{1}{(1-\beta t)^{\alpha_1}} \cdot \frac{1}{(1-\beta t)^{\alpha_2}} = \frac{1}{(1-\beta t)^{\alpha_1+\alpha_2}}. \end{aligned}$$

This is the MGF of a $\Gamma(\alpha_1 + \alpha_2, \beta)$ RV. So if $X \sim \Gamma(\alpha_1, \beta)$ and $Y \sim \Gamma(\alpha_2, \beta)$ are independent, then $U = X + Y \sim \Gamma(\alpha_1 + \alpha_2, \beta)$.

Theorem (6.3)

Let $Y_1, \ldots, Y_n$ be independent normally distributed random variables with $E[Y_i] = \mu_i$ and $V[Y_i] = \sigma_i^2$, for $i = 1, \ldots, n$, and let $a_1, \ldots, a_n$ be constants. If

$$U = \sum_{i=1}^{n} a_i Y_i,$$

then U is a normally distributed random variable with

$$E[U] = \sum_{i=1}^{n} a_i \mu_i \qquad \text{and} \qquad V[U] = \sum_{i=1}^{n} a_i^2 \sigma_i^2.$$

Sketch of Proof:

The normally distributed RV Y_i has the MGF

$$m_{Y_i}(t) = e^{\left(\mu_i t + \frac{\sigma_i^2 t^2}{2}\right)}$$

for each $i = 1, \ldots, n$.

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The normally distributed RV Y_i has the MGF

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By uniqueness of MGF, U is a normally distributed random variable with

$$E[U] = \sum_{i=1}^{n} a_i \mu_i \qquad \text{and} \qquad V[U] = \sum_{i=1}^{n} a_i^2 \sigma_i^2.$$

□

Theorem (6.4)

If $Z_1, \ldots, Z_n$ are independent standard normal RVs, then $U = Z_1^2 + \cdots + Z_n^2$ has the distribution $\chi^2[n]$. (Same as $\Gamma\left(\frac{n}{2}, 2\right)$.)

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Suppose that Y_1 and Y_2 are continuous random variables with joint density function $f_{Y_1,Y_2}(y_1, y_2)$ and that for all (y_1, y_2), such that $f_{Y_1,Y_2}(y_1, y_2) > 0$,

$$u_1 = h_1(y_1, y_2) \quad \text{and} \quad u_2 = h_2(y_1, y_2)$$

is a one-to-one transformation from (y_1, y_2) to (u_1, u_2) with inverse

$$y_1 = h_1^{-1}(u_1, u_2) \quad \text{and} \quad y_2 = h_2^{-1}(u_1, u_2).$$

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If $h_1^{-1}(u_1, u_2)$ and $h_2^{-1}(u_1, u_2)$ have continuous partial derivatives with respect to u_1 and u_2 and the *Jacobian*

$$J = \det \begin{bmatrix} \partial h_1^{-1}/\partial u_1 & \partial h_1^{-1}/\partial u_2 \\ \partial h_2^{-1}/\partial u_1 & \partial h_2^{-1}/\partial u_2 \end{bmatrix} = \frac{\partial h_1^{-1}}{\partial u_1}\frac{\partial h_2^{-1}}{\partial u_2} - \frac{\partial h_2^{-1}}{\partial u_1}\frac{\partial h_1^{-1}}{\partial u_2} \neq 0,$$

then the joint density of U_1 and U_2 is

$$f_{U_1,U_2}(u_1, u_2) = f_{Y_1,Y_2}\left(h_1^{-1}(u_1, u_2), h_2^{-1}(u_1, u_2)\right)|J|,$$

where $|J|$ is the absolute value of J.

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Caution:

Be sure that the bivariate transformation $u_1 = h_1(y_1, y_2)$, $u_2 = h_2(y_1, y_2)$ is a one-to-one transformation for all (y_1, y_2) such that $f_{Y_1,Y_2}(y_1, y_2) > 0$.

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Let's use this method for the following example:

Example 6.13

Let Y_1 and Y_2 be independent standard normal random variables. If $U_1 = Y_1 + Y_2$ and $U_2 = Y_1 - Y_2$, then what is the joint density of U_1 and U_2?

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The density functions for Y_1 and Y_2 are

$$f_1(y_1) = \frac{e^{-\frac{1}{2}y_1^2}}{\sqrt{2\pi}}, \quad f_2(y_2) = \frac{e^{-\frac{1}{2}y_2^2}}{\sqrt{2\pi}}, \quad \begin{array}{l} -\infty < y_1 < \infty, \\ -\infty < y_2 < \infty, \end{array}$$

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and the independence of Y_1 and Y_2 implies that their joint density is

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{2\pi} e^{-\frac{1}{2}(y_1^2+y_2^2)}, \qquad \begin{array}{c} -\infty < y_1 < \infty, \\ -\infty < y_2 < \infty. \end{array}$$

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In this case $f_{Y_1,Y_2}(y_1, y_2) > 0$ for all $-\infty < y_1 < \infty$ and $-\infty < y2 < \infty$.

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In this case $f_{Y_1,Y_2}(y_1, y_2) > 0$ for all $-\infty < y_1 < \infty$ and $-\infty < y2 < \infty$. We are interested in the transformation

$$u_1 = y_1 + y_2 = h_1(y_1, y_2) \quad \text{and} \quad u_2 = y_1 - y_2 = h_2(y_1, y_2),$$

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Because $\dfrac{\partial h_1^{-1}}{\partial u_1} = \dfrac{1}{2}$, $\dfrac{\partial h_1^{-1}}{\partial u_2} = \dfrac{1}{2}$, $\dfrac{\partial h_2^{-1}}{\partial u_1} = \dfrac{1}{2}$, and $\dfrac{\partial h_2^{-1}}{\partial u_2} = -\dfrac{1}{2}$, the Jacobian of this transformation is

Solution: (continued)

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Solution: (continued)

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Notice that U_1 and U_2 are *independent* and normally distributed, both with mean 0 and variance 2.

Solution: (continued)

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Notice that U_1 and U_2 are *independent* and normally distributed, both with mean 0 and variance 2. The extra information provided by the joint distribution of U_1 and U_2 is that the two variables are independent!

The k-variate Transformation

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If $Y_1, \ldots, Y_k$ are jointly continuous random variables and

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then there is a result analogous to the bivariate case that can be used to find the joint density of $U_1, \ldots, U_k$.

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$$\begin{aligned} g_{(j)(k)}(y_j, y_k) =& \frac{n!}{(j-1)!(k-1-j)!(n-k)!}[F(y_j)]^{j-1} \\ & \times [F(y_k) - F(y_j)]^{k-1-j} \times [1-F(y_k)]^{n-k} f(y_j) f(y_k). \end{aligned}$$

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We will look at the simplest cases $Y_{(n)}$ and $Y_{(1)}$.

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Exercise: (steps to find distributions of the minimum)

Let $Y_1, \ldots, Y_n$ be independent identically distributed continuous random variables with common distribution function $F(y)$ and common density function $f(y)$. Let $Y_m = \min\{Y_1, \ldots, Y_n\}$.

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(4) Find the probability density function f_m of Y_m in terms of F and f.

Solutions:

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The joint distribution of amount of pollutant emitted from a smokestack without a cleaning device (Y_1) and a similar smokestack with a cleaning device (Y_2) is

$$f(y_1, y_2) = \begin{cases} 1 & 0 \le y_1 \le 2, 0 \le y \le 1 \text{ and } 2y_2 \le y_1, \\ 0 & \text{elsewhere.} \end{cases}$$

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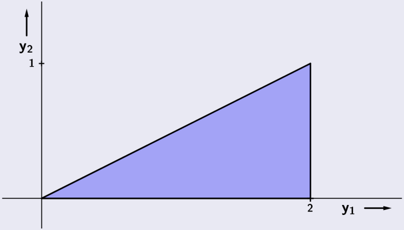

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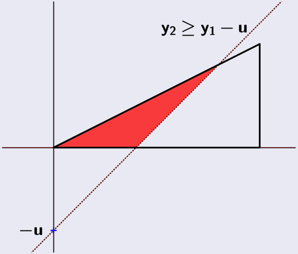

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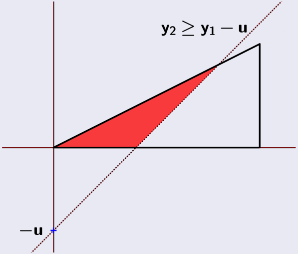
How does the picture change depending on the value of u?



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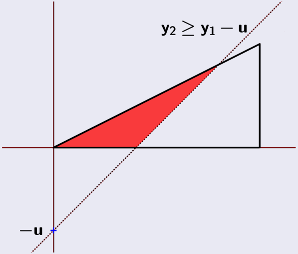

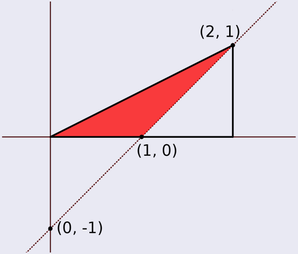

This picture shows that $u = 1$ is the "transition".

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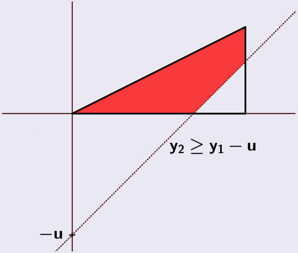

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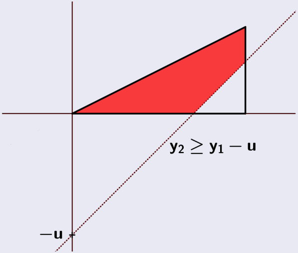
The area of the region is 1 minus the area of the small triangle.



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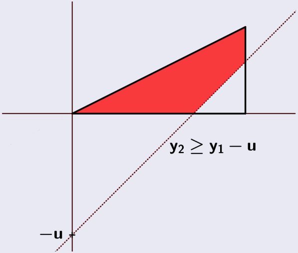
If $1 \leq u \leq 2$, this is how the region looks:
The area of the region is 1 minus the area of the small triangle.
The area of the shaded region, as a function of u, is the PDF of U.



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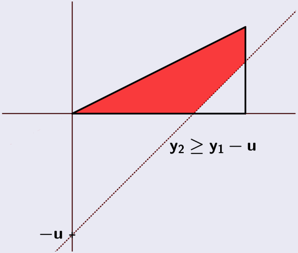

Example 6.4

Let Y have probability density function given by

$$f_Y(y) = \begin{cases} \frac{y+1}{2} & -1 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find the density function for $U = Y^2$.

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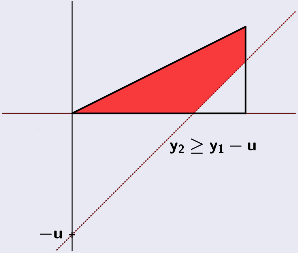

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Find the density function for $U = Y^2$.

Answer: $f_U(u) = \begin{cases} \frac{1}{2\sqrt{u}} & 0 < u \leq 1, \\ 0 & \text{elsewhere.} \end{cases}$

End of Chapter 6

Chapter 7

Sampling Distributions and the Central Limit Theorem

Sampling Distributions related to the Normal Distribution

Theorem (7.1)

Let $Y_1, \ldots, Y_n$ be a random sample of size n from a normal distribution with mean μ and variance σ^2.

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Because $Y_1, \ldots, Y_n$ is a random sample from a normal distribution with mean μ and variance σ^2, $Y_i, i = 1, \ldots, n$, are independent, normally distributed variables, with $E(Y_i) = \mu$ and $V(Y_i) = \sigma^2$.

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where $a_i = 1/n$, $i = 1, \ldots, n$.

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where $a_i = 1/n$, $i = 1, \ldots, n$. Thus, $\overline{Y}$ is a linear combination of $Y_1, \ldots, Y_n$.

Proof: (continued)

By Theorem 6.3, we conclude that $\overline{Y}$ is normally distributed with

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Remark:

Under the conditions of Theorem 7.1, $\overline{Y}$ is normally distributed with mean $\mu_{\overline{Y}} = \mu$ and variance $\sigma^2_{\overline{Y}} = \sigma^2/n$.

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$$E[\overline{Y}] = E\left[\frac{Y_1}{n} + \cdots + \frac{Y_n}{n}\right] = \underbrace{\frac{\mu}{n} + \cdots + \frac{\mu}{n}}_{n \text{ times}} = \mu,$$

and

$$V[\overline{Y}] = V\left[\frac{Y_1}{n} + \cdots + \frac{Y_n}{n}\right] = \underbrace{\frac{\sigma^2}{n^2} + \cdots + \frac{\sigma^2}{n^2}}_{n \text{ times}} = \frac{\sigma^2}{n}.$$

Remark:

Under the conditions of Theorem 7.1, $\overline{Y}$ is normally distributed with mean $\mu_{\overline{Y}} = \mu$ and variance $\sigma^2_{\overline{Y}} = \sigma^2/n$. It follows that

$$Z = \frac{\overline{Y} - \mu_{\overline{Y}}}{\sigma_{\overline{Y}}} = \frac{\overline{Y} - \mu}{\sigma/\sqrt{n}}$$

has the standard normal distribution.

Remarks:

- Notice that the variance of each of the random variables $Y_1, \ldots, Y_n$ is σ^2 and that of the sampling distribution of the random variable $\overline{Y}$ is σ^2/n.

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- Notice that the variance of each of the random variables $Y_1, \ldots, Y_n$ is σ^2 and that of the sampling distribution of the random variable $\overline{Y}$ is ${\sigma^2}/{n}$.
- With $\overline{Y}$ as in Theorem 7.1, it follows that
$$Z = \frac{\overline{Y} - \mu_{\overline{Y}}}{\sigma_{\overline{Y}}} = \frac{\overline{Y} - \mu}{\sigma/\sqrt{n}} = \sqrt{n}\left(\frac{\overline{Y} - \mu}{\sigma}\right)$$
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Remarks:

- Notice that the variance of each of the random variables $Y_1, \ldots, Y_n$ is σ^2 and that of the sampling distribution of the random variable $\overline{Y}$ is ${\sigma^2}/{n}$.
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has a standard normal distribution.

Example 7.2:

A bottling machine can be regulated so that it discharges an average of μ ounces per bottle. It has been observed that the amount of fill dispensed by the machine is normally distributed with $\sigma = 1.0$ ounce. A sample of $n = 9$ filled bottles is randomly selected from the output of the machine on a given day (all bottled with the same machine setting), and the ounces of fill are measured for each. Find the probability that the sample mean will be within .3 ounce of the true mean μ for the chosen machine setting.

Solution:

Solution:

If $Y_1, \ldots, Y_9$ denote the ounces of fill to be observed, then we know that the Y_is are normally distributed with mean μ and variance $\sigma^2 = 1$ for $i = 1, \ldots, 9$.

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If $Y_1, \ldots, Y_9$ denote the ounces of fill to be observed, then we know that the Y_is are normally distributed with mean μ and variance $\sigma^2 = 1$ for $i = 1, \ldots, 9$. Therefore, by Theorem 7.1, Y possesses a normal sampling distribution with mean $\mu_Y = \mu$ and variance $\sigma_Y^2 = {}^{\sigma}\!/\!_{\sqrt{n}} = {}^{1}\!/\!_{9}$.

Solution:

If $Y_1, \ldots, Y_9$ denote the ounces of fill to be observed, then we know that the Y_is are normally distributed with mean μ and variance $\sigma^2 = 1$ for $i = 1, \ldots, 9$. Therefore, by Theorem 7.1, Y possesses a normal sampling distribution with mean $\mu_Y = \mu$ and variance $\sigma_Y^2 = {}^{\sigma}/_{\sqrt{n}} = {}^{1}/_{9}$.

We want to find

$$P(|\overline{Y} - \mu| \leq 0.3) = P(-0.3 \leq \overline{Y} - \mu \leq 0.3)$$

Solution:

If $Y_1, \ldots, Y_9$ denote the ounces of fill to be observed, then we know that the Y_is are normally distributed with mean μ and variance $\sigma^2 = 1$ for $i = 1, \ldots, 9$. Therefore, by Theorem 7.1, Y possesses a normal sampling distribution with mean $\mu_Y = \mu$ and variance $\sigma_Y^2 = {}^{\sigma}/_{\sqrt{n}} = {}^{1}/_{9}$.
We want to find

$$\begin{aligned} P(|\overline{Y} - \mu| \leq 0.3) &= P(-0.3 \leq \overline{Y} - \mu \leq 0.3) \\ &= P\left(-\frac{0.3}{\sigma/\sqrt{n}} \leq \frac{\overline{Y} - \mu}{\sigma/\sqrt{n}} \leq \frac{0.3}{\sigma/\sqrt{n}}\right). \end{aligned}$$

Solution:

If $Y_1, \ldots, Y_9$ denote the ounces of fill to be observed, then we know that the Y_is are normally distributed with mean μ and variance $\sigma^2 = 1$ for $i = 1, \ldots, 9$. Therefore, by Theorem 7.1, Y possesses a normal sampling distribution with mean $\mu_Y = \mu$ and variance $\sigma_Y^2 = \sigma/\sqrt{n} = 1/9$.

We want to find

$$P(|\overline{Y} - \mu| \leq 0.3) = P(-0.3 \leq \overline{Y} - \mu \leq 0.3)$$

$$= P\left(-\frac{0.3}{\sigma/\sqrt{n}} \leq \frac{\overline{Y} - \mu}{\sigma/\sqrt{n}} \leq \frac{0.3}{\sigma/\sqrt{n}}\right).$$

Because $\dfrac{\overline{Y} - \mu_{\overline{Y}}}{\sigma_{\overline{Y}}} = \dfrac{\overline{Y} - \mu}{\sigma/\sqrt{n}}$ has a standard normal distribution, it follows that

$$P(|\overline{Y} - \mu| \leq 0.3) = P\left(-\frac{0.3}{1/\sqrt{9}} \leq Z \leq \frac{0.3}{1/\sqrt{9}}\right)$$

Solution:

If $Y_1, \ldots, Y_9$ denote the ounces of fill to be observed, then we know that the Y_is are normally distributed with mean μ and variance $\sigma^2 = 1$ for $i = 1, \ldots, 9$. Therefore, by Theorem 7.1, Y possesses a normal sampling distribution with mean $\mu_Y = \mu$ and variance $\sigma^2_Y = {}^{\sigma}/_{\sqrt{n}} = {}^{1}/_{9}$.
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$$\begin{aligned} P(|\overline{Y} - \mu| \leq 0.3) &= P(-0.3 \leq \overline{Y} - \mu \leq 0.3) \\ &= P\left(-\frac{0.3}{\sigma/\sqrt{n}} \leq \frac{\overline{Y} - \mu}{\sigma/\sqrt{n}} \leq \frac{0.3}{\sigma/\sqrt{n}}\right). \end{aligned}$$

Because $\dfrac{\overline{Y} - \mu_{\overline{Y}}}{\sigma_{\overline{Y}}} = \dfrac{\overline{Y} - \mu}{\sigma/\sqrt{n}}$ has a standard normal distribution, it follows that

$$P(|\overline{Y} - \mu| \leq 0.3) = P\left(-\frac{0.3}{1/\sqrt{9}} \leq Z \leq \frac{0.3}{1/\sqrt{9}}\right) = P(-0.9 \leq Z \leq 0.9).$$

Solution:

If $Y_1, \ldots, Y_9$ denote the ounces of fill to be observed, then we know that the Y_is are normally distributed with mean μ and variance $\sigma^2 = 1$ for $i = 1, \ldots, 9$. Therefore, by Theorem 7.1, Y possesses a normal sampling distribution with mean $\mu_Y = \mu$ and variance $\sigma_Y^2 = {}^{\sigma}/\sqrt{n} = {}^{1}/{9}$.
We want to find

$$P(|\overline{Y} - \mu| \leq 0.3) = P(-0.3 \leq \overline{Y} - \mu \leq 0.3)$$
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Using Table 4, Appendix 3, we find

$$P(-0.9 \leq Z \leq 0.9) = 1 - 2P(Z > 0.9)$$

Solution:

If $Y_1, \ldots, Y_9$ denote the ounces of fill to be observed, then we know that the Y_is are normally distributed with mean μ and variance $\sigma^2 = 1$ for $i = 1, \ldots, 9$. Therefore, by Theorem 7.1, Y possesses a normal sampling distribution with mean $\mu_Y = \mu$ and variance $\sigma^2_Y = {}^{\sigma}/_{\sqrt{n}} = {}^{1}/_{9}$.

We want to find

$$P(|\overline{Y} - \mu| \leq 0.3) = P(-0.3 \leq \overline{Y} - \mu \leq 0.3)$$

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Because $\dfrac{\overline{Y} - \mu_{\overline{Y}}}{\sigma_{\overline{Y}}} = \dfrac{\overline{Y} - \mu}{\sigma/\sqrt{n}}$ has a standard normal distribution, it follows that

$$P(|\overline{Y} - \mu| \leq 0.3) = P\left(-\frac{0.3}{1/\sqrt{9}} \leq Z \leq \frac{0.3}{1/\sqrt{9}}\right) = P(-0.9 \leq Z \leq 0.9).$$

Using Table 4, Appendix 3, we find

$$P(-0.9 \leq Z \leq 0.9) = 1 - 2P(Z > 0.9) = 1 - 2(0.1841) = 0.6318.$$

Solution:

If $Y_1, \ldots, Y_9$ denote the ounces of fill to be observed, then we know that the Y_is are normally distributed with mean μ and variance $\sigma^2 = 1$ for $i = 1, \ldots, 9$. Therefore, by Theorem 7.1, Y possesses a normal sampling distribution with mean $\mu_Y = \mu$ and variance $\sigma^2_Y = \sigma/\sqrt{n} = 1/9$.

We want to find

$$P(|\overline{Y} - \mu| \leq 0.3) = P(-0.3 \leq \overline{Y} - \mu \leq 0.3)$$
$$= P\left(-\frac{0.3}{\sigma/\sqrt{n}} \leq \frac{\overline{Y} - \mu}{\sigma/\sqrt{n}} \leq \frac{0.3}{\sigma/\sqrt{n}}\right).$$

Because $\dfrac{\overline{Y} - \mu_{\overline{Y}}}{\sigma_{\overline{Y}}} = \dfrac{\overline{Y} - \mu}{\sigma/\sqrt{n}}$ has a standard normal distribution, it follows that

$$P(|\overline{Y} - \mu| \leq 0.3) = P\left(-\frac{0.3}{1/\sqrt{9}} \leq Z \leq \frac{0.3}{1/\sqrt{9}}\right) = P(-0.9 \leq Z \leq 0.9).$$

Using Table 4, Appendix 3, we find

$$P(-0.9 \leq Z \leq 0.9) = 1 - 2P(Z > 0.9) = 1 - 2(0.1841) = 0.6318.$$

Thus, the probability is only .6318 that the sample mean will be within .3 ounce of the true population mean.

Example 7.3:

Refer to Example 7.2. How many observations should be included in the sample if we wish $\overline{Y}$ to be within .3 ounce of μ with probability .95?

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Divide each term of the inequality by $\sigma_{\overline{Y}} = {}^{\sigma}/_{\sqrt{n}}$ to get

$$P\left(-\frac{0.3}{\sigma/\sqrt{n}} \leq \frac{\overline{Y} - \mu}{\sigma/\sqrt{n}} \leq \frac{0.3}{\sigma/\sqrt{n}}\right) = P(-0.3\sqrt{n} \leq Z \leq 0.3\sqrt{n}) = 0.95.$$

(Recall that $\sigma = 1$).

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(Recall that $\sigma = 1$). But using Table 4, Appendix 3, we obtain $P(-1.96 \leq Z \leq 1.96) = 0.95$.

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(Recall that $\sigma = 1$). But using Table 4, Appendix 3, we obtain $P(-1.96 \leq Z \leq 1.96) = 0.95$. It must follow that

$$0.3\sqrt{n} = 1.96$$

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$$0.3\sqrt{n} = 1.96 \implies n = \left(\frac{1.96}{0.3}\right)^2$$

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(Recall that $\sigma = 1$). But using Table 4, Appendix 3, we obtain $P(-1.96 \leq Z \leq 1.96) = 0.95$. It must follow that

$$0.3\sqrt{n} = 1.96 \implies n = \left(\frac{1.96}{0.3}\right)^2 \approx 42.68.$$

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Refer to Example 7.2. How many observations should be included in the sample if we wish $\overline{Y}$ to be within .3 ounce of μ with probability .95?

Solution:

Now we want

$$P(|\overline{Y} - \mu| \leq 0.3) = P(-0.3 \leq \overline{Y} - \mu \leq 0.3) = 0.95.$$

Divide each term of the inequality by $\sigma_{\overline{Y}} = {}^{\sigma}/_{\sqrt{n}}$ to get

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Practically, it is impossible to take a sample of size 42.68.

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Refer to Example 7.2. How many observations should be included in the sample if we wish $\overline{Y}$ to be within .3 ounce of μ with probability .95?

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Now we want

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(Recall that $\sigma = 1$). But using Table 4, Appendix 3, we obtain $P(-1.96 \leq Z \leq 1.96) = 0.95$. It must follow that

$$0.3\sqrt{n} = 1.96 \implies n = \left(\frac{1.96}{0.3}\right)^2 \approx 42.68.$$

Practically, it is impossible to take a sample of size 42.68. Our solution indicates that a sample of size 42 is not quite large enough to reach our objective.

Example 7.3:

Refer to Example 7.2. How many observations should be included in the sample if we wish $\overline{Y}$ to be within .3 ounce of μ with probability .95?

Solution:

Now we want

$$P(|\overline{Y} - \mu| \leq 0.3) = P(-0.3 \leq \overline{Y} - \mu \leq 0.3) = 0.95.$$

Divide each term of the inequality by $\sigma_{\overline{Y}} = \sigma/\sqrt{n}$ to get

$$P\left(-\frac{0.3}{\sigma/\sqrt{n}} \leq \frac{\overline{Y} - \mu}{\sigma/\sqrt{n}} \leq \frac{0.3}{\sigma/\sqrt{n}}\right) = P(-0.3\sqrt{n} \leq Z \leq 0.3\sqrt{n}) = 0.95.$$

(Recall that $\sigma = 1$). But using Table 4, Appendix 3, we obtain $P(-1.96 \leq Z \leq 1.96) = 0.95$. It must follow that

$$0.3\sqrt{n} = 1.96 \implies n = \left(\frac{1.96}{0.3}\right)^2 \approx 42.68.$$

Practically, it is impossible to take a sample of size 42.68. Our solution indicates that a sample of size 42 is not quite large enough to reach our objective. If $n = 43$, $P(|Y - \mu| \leq 0.3)$ slightly exceeds 0.95.

Theorem (7.2)

Let $Y_1, \ldots, Y_n$ be as in Theorem 7.1. Then $Z_i = \dfrac{Y_i - \mu}{\sigma}$ are independent standard normal random variables, $i = 1, \ldots, n$, and

$$\sum_{i=1}^{n} Z_i^2 = \sum_{i=1}^{n} \left(\frac{Y_i - \mu}{\sigma} \right)^2$$

has a χ^2 distribution with n degrees of freedom.

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Proof.

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has a χ^2 distribution with n degrees of freedom.

Proof.

Because $Y_1, \ldots, Y_n$ is a random sample from a normal distribution with mean μ and variance σ^2, $Z_i = \dfrac{Y_i - \mu}{\sigma}$ has a standard normal distribution for $i = 1, \ldots, n$.

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$$\sum_{i=1}^{n} Z_i^2 = \sum_{i=1}^{n} \left(\frac{Y_i - \mu}{\sigma} \right)^2$$

has a χ^2 *distribution with* n *degrees of freedom.*

Proof.

Because $Y_1, \ldots, Y_n$ is a random sample from a normal distribution with mean μ and variance σ^2, $Z_i = \dfrac{Y_i - \mu}{\sigma}$ has a standard normal distribution for $i = 1, \ldots, n$. Further, the random variables Z_i are independent as the random variables Y_i are independent, $i = 1, \ldots, n$.

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Let $Y_1, \ldots, Y_n$ be as in Theorem 7.1. Then $Z_i = \dfrac{Y_i - \mu}{\sigma}$ are independent standard normal random variables, $i = 1, \ldots, n$, and

$$\sum_{i=1}^{n} Z_i^2 = \sum_{i=1}^{n} \left(\frac{Y_i - \mu}{\sigma} \right)^2$$

has a χ^2 distribution with n degrees of freedom.

Proof.

Because $Y_1, \ldots, Y_n$ is a random sample from a normal distribution with mean μ and variance σ^2, $Z_i = \dfrac{Y_i - \mu}{\sigma}$ has a standard normal distribution for $i = 1, \ldots, n$. Further, the random variables Z_i are independent as the random variables Y_i are independent, $i = 1, \ldots, n$. It follows directly from Theorem 6.4 that $\sum_{i=1}^{n} Z_i^2$ has the distribution $\chi^2[n]$. $\square$

Remark:

From Table 6, Appendix 3, we can find values χ^2_α so that $P(\chi^2 > \chi^2_\alpha) = \alpha$, that is, $P(\chi^2 \leq \chi^2_\alpha) = 1 - \alpha$.

Remark:

From Table 6, Appendix 3, we can find values χ^2_α so that $P(\chi^2 > \chi^2_\alpha) = \alpha$, that is, $P(\chi^2 \leq \chi^2_\alpha) = 1 - \alpha$. Thus χ^2_α is the $(1-\alpha)$ quantile of the χ^2 RV.

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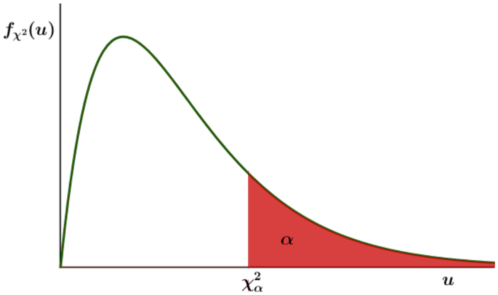

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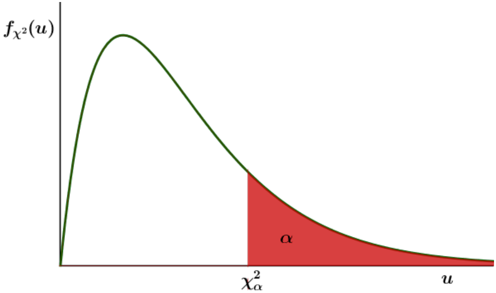

The following example illustrates the combined use of Theorem 7.2 and the χ^2 tables.

Example 7.4:

If $Z_1, \ldots, Z_6$ denotes a random sample from the standard normal distribution, find a number b such that

$$P\left(\sum_{i=1}^{6} Z_i^2 \leq b\right) = 0.95.$$

Example 7.4:

If $Z_1, \ldots, Z_6$ denotes a random sample from the standard normal distribution, find a number b such that

$$P\left(\sum_{i=1}^{6} Z_i^2 \leq b\right) = 0.95.$$

Solution:

Example 7.4:

If $Z_1, \ldots, Z_6$ denotes a random sample from the standard normal distribution, find a number b such that

$$P\left(\sum_{i=1}^{6} Z_i^2 \leq b\right) = 0.95.$$

Solution:

By Theorem 7.2, $\sum_{i=1}^{6} Z_i^2$ has the distribution $\chi^2[6]$.

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If $Z_1, \ldots, Z_6$ denotes a random sample from the standard normal distribution, find a number b such that

$$P\left(\sum_{i=1}^{6} Z_i^2 \leq b\right) = 0.95.$$

Solution:

By Theorem 7.2, $\sum_{i=1}^{6} Z_i^2$ has the distribution $\chi^2[6]$. Looking at Table 6, Appendix 3, in the row headed 6 df and the column headed $\chi^2_{.05}$, we see the number 12.5916.

Example 7.4:

If $Z_1, \ldots, Z_6$ denotes a random sample from the standard normal distribution, find a number b such that

$$P\left(\sum_{i=1}^{6} Z_i^2 \leq b\right) = 0.95.$$

Solution:

By Theorem 7.2, $\sum_{i=1}^{6} Z_i^2$ has the distribution $\chi^2[6]$. Looking at Table 6, Appendix 3, in the row headed 6 df and the column headed $\chi^2_{.05}$, we see the number 12.5916. Thus

$$P\left(\sum_{i=1}^{6} Z_i^2 > 12.5916\right) = 0.05$$

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If $Z_1, \ldots, Z_6$ denotes a random sample from the standard normal distribution, find a number b such that

$$P\left(\sum_{i=1}^{6} Z_i^2 \leq b\right) = 0.95.$$

Solution:

By Theorem 7.2, $\sum_{i=1}^{6} Z_i^2$ has the distribution $\chi^2[6]$. Looking at Table 6, Appendix 3, in the row headed 6 df and the column headed $\chi^2_{.05}$, we see the number 12.5916. Thus

$$P\left(\sum_{i=1}^{6} Z_i^2 > 12.5916\right) = 0.05 \iff P\left(\sum_{i=1}^{6} Z_i^2 \leq 12.5916\right) = 0.95,$$

Example 7.4:

If $Z_1, \ldots, Z_6$ denotes a random sample from the standard normal distribution, find a number b such that

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Solution:

By Theorem 7.2, $\sum_{i=1}^{6} Z_i^2$ has the distribution $\chi^2[6]$. Looking at Table 6, Appendix 3, in the row headed 6 df and the column headed $\chi^2_{.05}$, we see the number 12.5916. Thus

$$P\left(\sum_{i=1}^{6} Z_i^2 > 12.5916\right) = 0.05 \iff P\left(\sum_{i=1}^{6} Z_i^2 \leq 12.5916\right) = 0.95,$$

and $b = 12.5916$ is the .95 quantile (95^{th} percentile) of the sum of the squares of six independent standard normal random variables.

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Let $Y_1, \ldots, Y_n$ be a random sample from a normal distribution with mean μ and variance σ^2. Then

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has the distribution $\chi^2[n-1]$. Also $\overline{Y}$ and S^2 are independent random variables.

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We will show that this quantity is equal to the square of a standard normal random variable; that is, it is a Z^2, which possesses the distribution $\chi^2[1]$.

Proof: (continued)

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Because $\overline{Y}$ is a function of only U_1 and S^2 is a function of only U_2, the independence of U_1, U_2 implies the independence of $\overline{Y}$ and S^2.

The t-Distribution

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Let Z be a standard normal random variable and let W be a $\chi^2[\nu]$-distributed variable. Then, if Z and W are independent, $T = \frac{Z}{\sqrt{W/\nu}}$ is said to have the t-distribution with ν degrees of freedom (or parameter ν).

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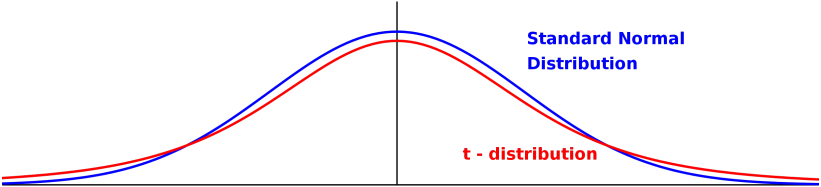
(b) Find the joint density of T and W, $f(t, w)$, by using $f(t, w) = f(t \mid w)f(w)$.

(c) Integrate over w to show that

$$f(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\Gamma\left(\frac{\nu}{2}\right)}\left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \qquad -\infty < t < \infty.$$


Standard Normal
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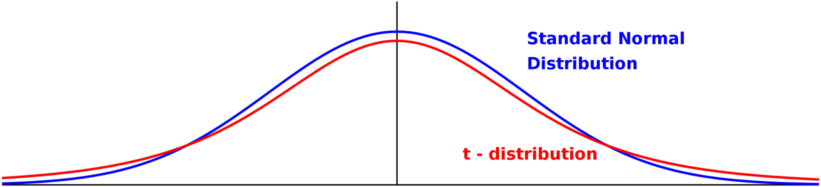
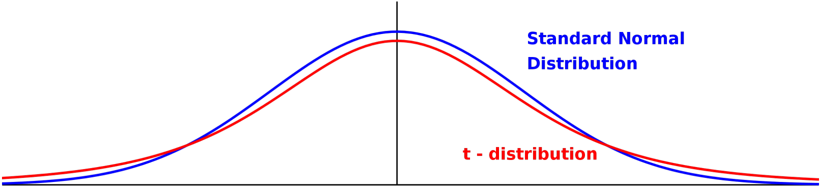


Table 5, Appendix 3 lists the values of t_α such that $P(T > t_\alpha)$.



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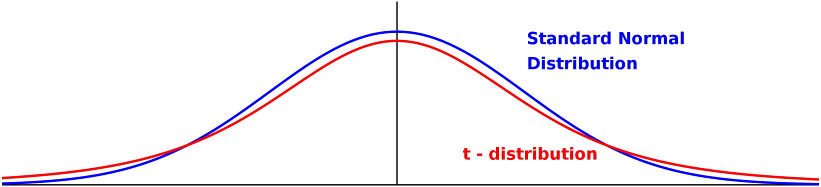
In general, $t_\alpha = \phi_{1-\alpha}$, the $(1-\alpha)$ quantile (the $100(1-\alpha)^{\text{th}}$ percentile) of a t-distributed RV.



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Example 7.6:

The tensile strength for a type of wire is normally distributed with unknown mean μ and unknown variance σ^2. Six pieces of wire were randomly selected from a large roll; Y_i, the tensile strength for portion i, is measured for $i = 1, \dots, 6$. The population mean μ and variance σ^2 can be estimated by $\overline{Y}$ and S^2, respectively. Because $\sigma^2_{\overline{Y}} = \sigma^2/n$, it follows that $\sigma^2_{\overline{Y}}$ can be estimated by σ^2/n. Find the approximate probability that $\overline{Y}$ will be within $2S/\sqrt{n}$ of the true population mean μ.

Solution:

Solution:

We want to find

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where T has a t-distribution with, in this case, $n - 1 = 5$ df. Table 5, Appendix 3 suggests that the upper-tail area to the right of 2.015 is 0.05. Hence $P(-2.015 \le T \le 2.015) = 0.9$, and the probability that $\overline{Y}$ will be within 2 estimated standard deviations of μ is slightly less than 0.9.

Solution:

We want to find

$$P\left(-\frac{2S}{\sqrt{n}} \le \overline{Y} - \mu \le \frac{2S}{\sqrt{n}}\right) = P\left(-2 \le \sqrt{n}\left(\frac{\overline{Y} - \mu}{S}\right) \le 2\right)$$
$$= P(-2 \le T \le 2),$$

where T has a t-distribution with, in this case, $n - 1 = 5$ df. Table 5, Appendix 3 suggests that the upper-tail area to the right of 2.015 is 0.05. Hence $P(-2.015 \le T \le 2.015) = 0.9$, and the probability that $\overline{Y}$ will be within 2 estimated standard deviations of μ is slightly less than 0.9.

Remark:

If σ^2 were known, the probability that $\overline{Y}$ will fll within $2\sigma_{\overline{Y}}$ of μ would be

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The F-Distribution

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The ratio $\dfrac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} = \dfrac{\sigma_2^2}{\sigma_1^2}\left(\dfrac{S_1^2}{S_2^2}\right)$ has the F-distribution with $n_1 - 1$ numerator degrees of freedom and $n_2 - 1$ denominator degrees of freedom.

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Definition (7.3)

Let W_1 and W_2 be independent χ^2-distributed random variables with ν_1 and ν_2 df, respectively. Then $F = \dfrac{W_1/\nu_1}{W_2/\nu_2}$ is said to have an F-distribution with ν_1 numerator degrees of freedom and ν_2 denominator degrees of freedom.

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(a) If W_2 is fixed at w_2, then $F = {}^{W_1}\!/_{c}$, where $c = {}^{w_2 \nu_1}\!/_{\nu_2}$. Find the conditional density of F for fixed $W_2 = w_2$.

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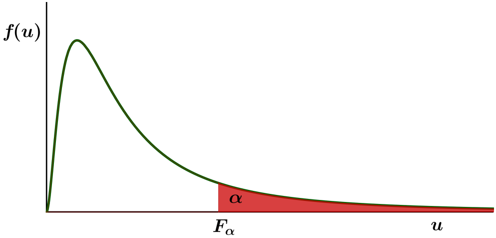
(b) Find the joint density of F and W_2.

(c) Integrate over w_2 to show that the probability density function of F – say, $g(y)$ – is given by

$$g(y) = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)\left({}^{\nu_1}\!/_{\nu_2}\right)^{\nu_1/2}}{\Gamma\left({}^{\nu_1}\!/_2\right)\Gamma\left({}^{\nu_2}\!/_2\right)} y^{\frac{\nu_1}{2}-1}\left(1+\frac{\nu_1 y}{\nu_2}\right)^{-\frac{\nu_1+\nu_2}{2}}, \quad 0 < y < \infty.$$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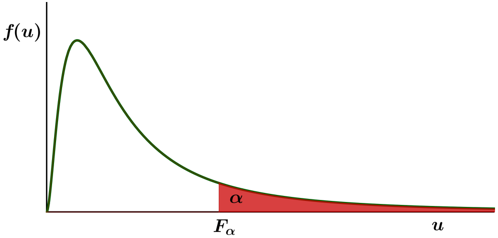
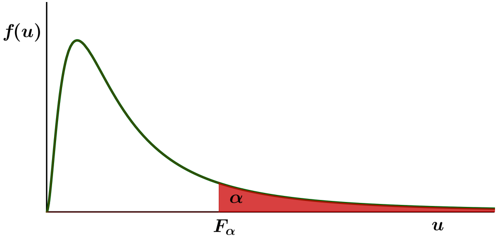


Table 7, Appendix 3 lists values of F_α such that $P(F > F_\alpha)$.



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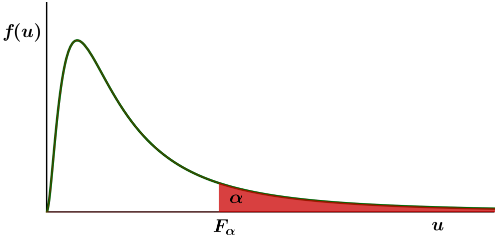
In general, $F_\alpha = \phi_{1-\alpha}$, the $(1-\alpha)$ quantile (the $100(1-\alpha)^{\text{th}}$ percentile) of an F-distributed RV.



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In general, $F_\alpha = \phi_{1-\alpha}$, the $(1-\alpha)$ quantile (the $100(1-\alpha)^{\text{th}}$ percentile) of an F-distributed RV.

Example 7.7:

If we take independent samples of size $n_1 = 6$ and $n_2 = 10$ from two normal populations with equal population variances, find the number b such that

$$P\left(\frac{S_1^2}{S_2^2} \leq b\right) = 0.95.$$

Solution:

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Because $n_1 = 6$ and $n_2 = 10$, and the population variances are equal,
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Solution:

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Solution:

Because $n_1 = 6$ and $n_2 = 10$, and the population variances are equal, $\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} = \frac{S_1^2}{S_2^2}$ has an F-distribution with $\nu_1 = n_1 - 1 = 5$ numerator degrees of freedom and $\nu_2 = n_2 - 1 = 9$ denominator degrees of freedom. Also,

$$P\left(\frac{S_1^2}{S_2^2} \leq b\right) = 1 - P\left(\frac{S_1^2}{S_2^2} > b\right).$$

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Because $n_1 = 6$ and $n_2 = 10$, and the population variances are equal, $\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} = \frac{S_1^2}{S_2^2}$ has an F-distribution with $\nu_1 = n_1 - 1 = 5$ numerator degrees of freedom and $\nu_2 = n_2 - 1 = 9$ denominator degrees of freedom. Also,

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Therefore, we want to find the number b cutting off an upper-tail area of 0.05 under the F density function with 5 numerator degrees of freedom and 9 denominator degrees of freedom.

Solution:

Because $n_1 = 6$ and $n_2 = 10$, and the population variances are equal, $\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} = \frac{S_1^2}{S_2^2}$ has an F-distribution with $\nu_1 = n_1 - 1 = 5$ numerator degrees of freedom and $\nu_2 = n_2 - 1 = 9$ denominator degrees of freedom. Also,

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Therefore, we want to find the number b cutting off an upper-tail area of 0.05 under the F density function with 5 numerator degrees of freedom and 9 denominator degrees of freedom. Looking in column 5 and row 9 in Table 7, Appendix 3, we see that the appropriate value of b is 3.48.

Solution:

Because $n_1 = 6$ and $n_2 = 10$, and the population variances are equal, $\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} = \frac{S_1^2}{S_2^2}$ has an F-distribution with $\nu_1 = n_1 - 1 = 5$ numerator degrees of freedom and $\nu_2 = n_2 - 1 = 9$ denominator degrees of freedom. Also,

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Remark:

Even when the population variances are equal, the probability that the ratio of the sample variances exceeds 3.48 is still 0.05 (assuming sample sizes of $n_1 = 6$ and $n_2 = 10$).

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Note: The formula $U_n = \dfrac{\overline{Y} - \mu}{\sigma/\sqrt{n}}$, where $\overline{Y} = \dfrac{1}{n}\displaystyle\sum_{i=1}^n Y_i$, guarantees that $E[U_n] = 0$.

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Remark: There are other senses of convergence, such as "weak convergence", "almost sure convergence", etc.

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- No specific distribution is mentioned. So if we don't apply the CLT, how can we do the problem?
- We are given the population standard deviation, which we need to apply the CLT for this exercise.

Solution:

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But U_{100} is approximately standard normal. Using the "95% rule" approximation, the answer is $P\left(|U_{100}| < 2\right) = 1 - 2(0.0228) \approx 95.4\%$.

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Let Z be a standard normal random variable and let W be a $\chi^2[\nu]$-distributed variable. Then, if Z and W are independent, $T = \dfrac{Z}{\sqrt{W/\nu}}$ is said to have the t-distribution with ν degrees of freedom (or parameter ν).

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Reason: "fatter tails".

Recall: The Central Limit Theorem

Theorem (7.4)

Let $Y_1, \ldots, Y_n$ be independent and identically distributed random variables with $E[Y_i] = \mu$ and $V[Y_i] = \sigma^2 < \infty$.

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(c) Limit definition of the exponential function.

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Let Y and $Y_1, Y_2, \ldots$ be random variables with moment-generating functions $m(t)$ and $m_1(t), m_2(t), \ldots,$ respectively. If $\lim_{n\to\infty} m_n(t) = m(t)$ for all real t, then the distribution function of Y_n converges to the distribution function of Y as $n \to \infty$.

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$$f(t) = \underbrace{f(0) + f'(0)\cdot t}_{\text{linear approximation to } f} + \underbrace{\frac{f''(\xi)}{2}\cdot t^2}_{\text{error term}}, \quad \text{where } 0 < \xi < t.$$

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Recall that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ and $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$,

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Proof: (continued)

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Recall that we “normalized” $Z_1 = \dfrac{Y_1 - \mu}{\sigma}$ so that it has mean 0 and variance 1.

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- You should be able to produce a correct statement of the CLT. Most importantly, using this, you should be able to do the problems of the form "Apply the CLT", even if the problem statements do not explicitly mention the CLT.

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Exercise 7.45:

Workers employed in a large service industry have an average wage of \$7.00 per hour with a standard deviation of \$0.50. The industry has 64 workers of a certain ethnic group. These workers have an average wage of \$6.90 per hour. Is it reasonable to assume that the wage rate of the ethnic group is equivalent to that of a random sample of workers from those employed in the service industry?

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Solution: (continued)

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Let $Y_1, \ldots, Y_5$ be a random sample of size 5 from a normal population with mean 0 and variance 1 and let $Y = \frac{1}{5}\sum_{i=1}^{5} Y_i$. What is the distribution of $W = \sum_{i=1}^{5} Y_i^2$? Why?

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<u>Answer:</u> Using MGFs, we can show that $W \sim \chi^2[5]$.

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By CLT, the braced expression above is standard normal for large n.

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Now we need to find n such that $P\left(|Z| \leq \frac{1}{10/\sqrt{n}}\right) = 0.99$.

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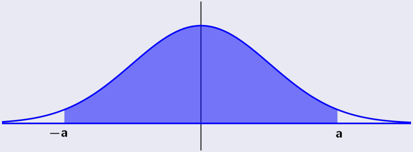

For which a is the shaded area $= 0.99$?

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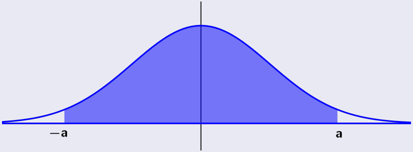

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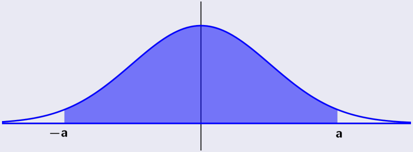

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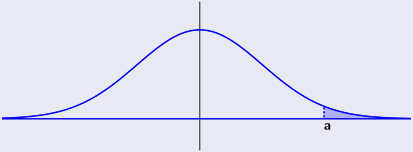

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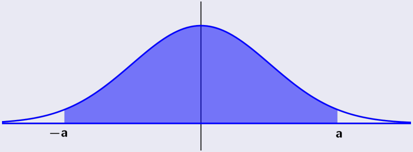

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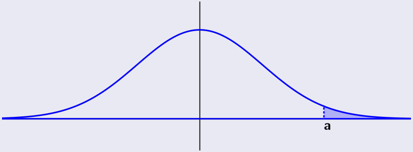

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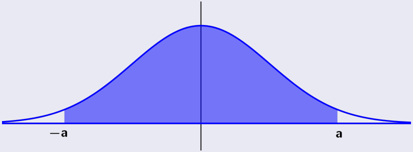

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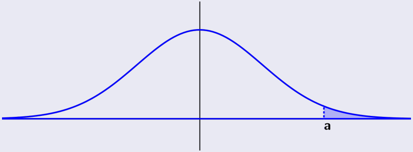

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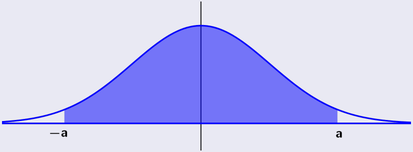

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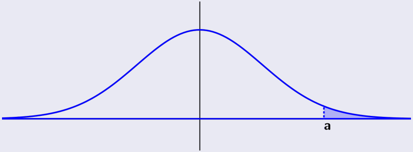

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$$\frac{1}{10/\sqrt{n}} \approx 2.575$$

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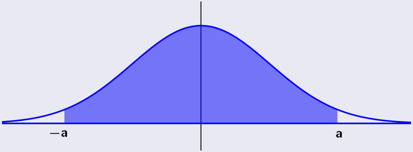

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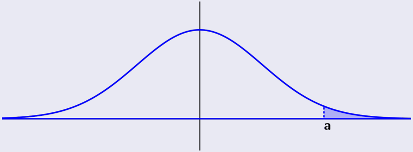

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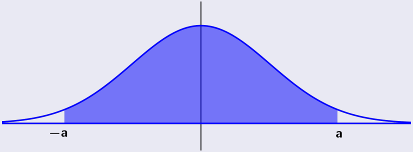

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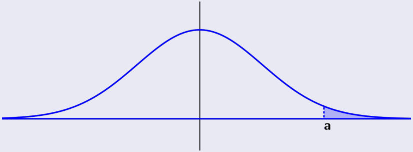

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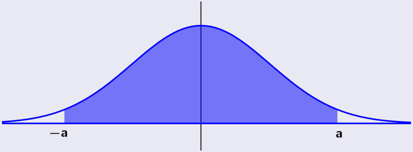

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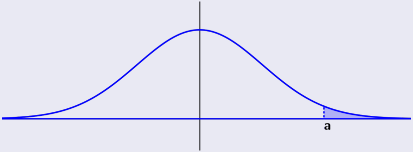

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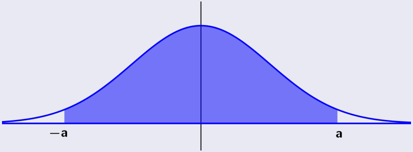

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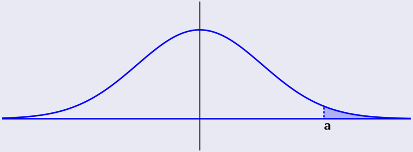

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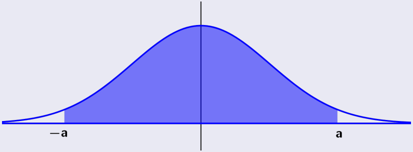

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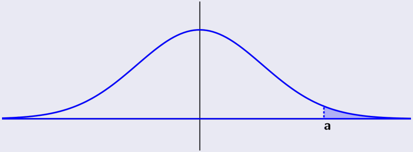

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Thus $n = 663$ is good enough.

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One-hour carbon monoxide concentrations in air samples from a large city average 12 ppm (parts per million) with standard deviation 9 ppm. Find the probability that the average concentration in 100 randomly selected samples will exceed 14 ppm.

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In such problems, CLT applies even though $Y_1, \ldots, Y_n$ are NOT normally distributed. What is normally distributed is $U_n = \dfrac{\overline{Y} - \mu}{\sigma/\sqrt{n}}$.

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$$Y = \sum_{i=1}^{n} X_i, \qquad \text{where } X_i = \begin{cases} 1 & \text{if the } i^{\text{th}} \text{ trial is success,} \\ 0 & \text{otherwise.} \end{cases}$$

The random variables X_i for $i = 1, \ldots, n$ are independent (because the trials are independent), and it is easy to show that $E[X_i] = p$ and $V[X_i] = p(1-p)$ for$i = 1, \ldots, n$.

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$$\frac{Y}{n} = \frac{1}{n} \sum_{i=1}^{n} X_i = \overline{X},$$

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Since $n = 25 > 13.5$, the normal approximation is indeed adequate.

Here is a comparison of the distributions $Y \sim \text{Bin}(25, 0.4)$ (histogram in blue) and the normal approximation $W \sim \mathcal{N}(10, 6)$ (green):

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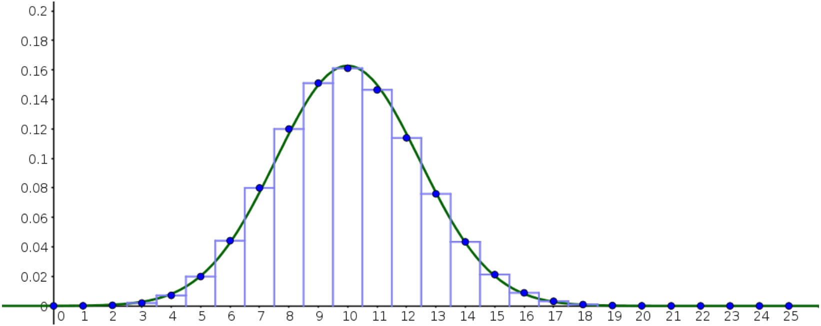

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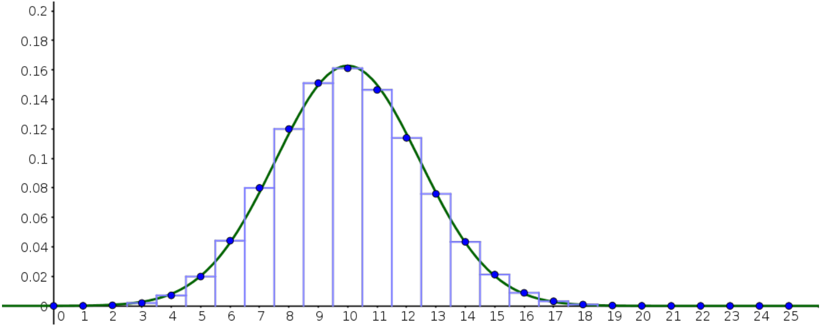

Note that

$$\mu_W = np = 25(0.4) = 10,$$

and

$$\sigma_W^2 = np(1-p) = 25(0.4)(0.6) = 6.$$

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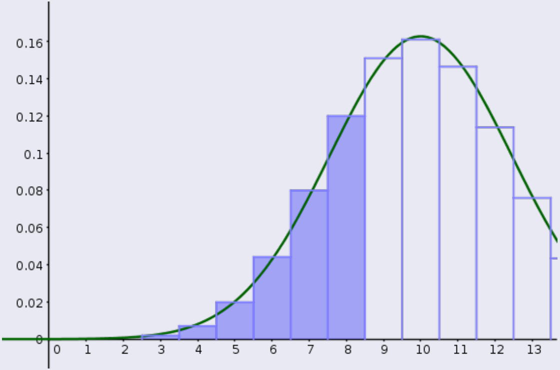

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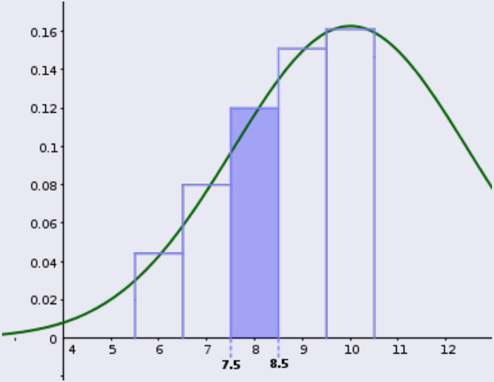

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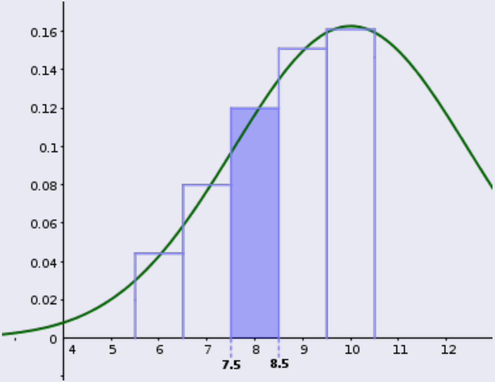

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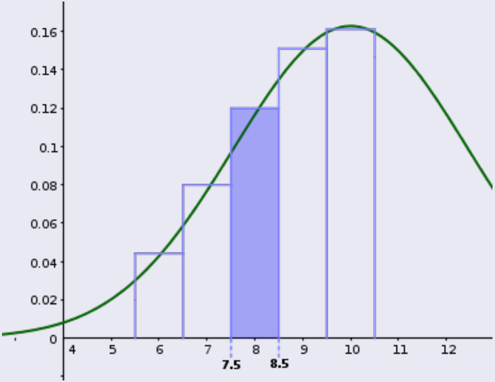

From the table, we find

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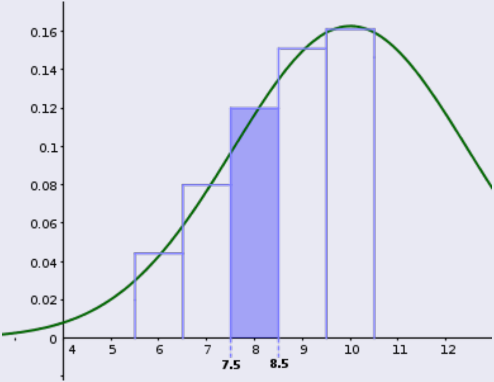

From the table, we find

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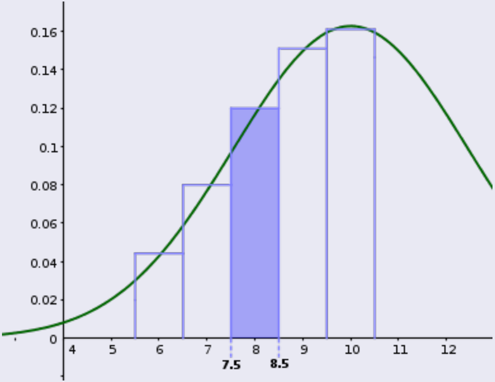
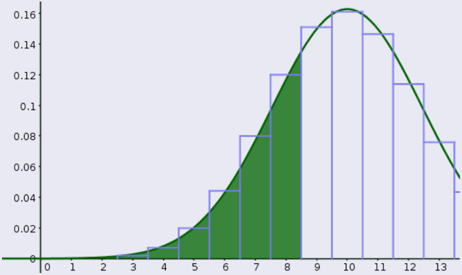


Now our normal approximation is $W \sim \mathcal{N}(10, 6)$. Looking at the picture, we need to find $P(Y \leq 8) \approx P(W \leq 8.5)$, and $P(Y = 8) \approx P(7.5 \leq W \leq 8.5)$; the half-integers accouting for the (obvious) correction.

Solution: (continued)



Thus

$$P(W \leq 8.5)$$

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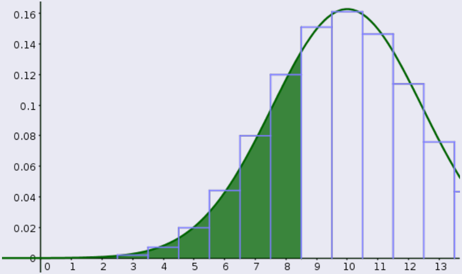
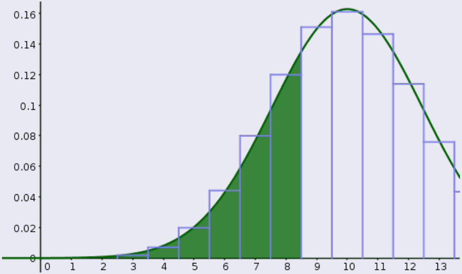


Thus

$$P(W \leq 8.5)$$
$$= P\left(\frac{W-10}{\sqrt{6}} \leq \frac{8.5-10}{\sqrt{6}}\right)$$

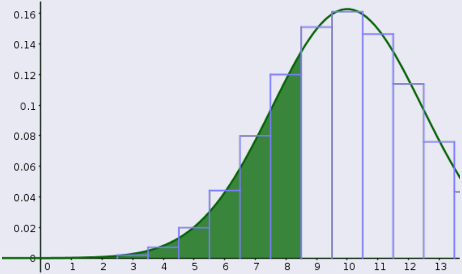
Solution: (continued)



Thus

$$
\begin{aligned}
&P(W \leq 8.5) \\
=&P\left(\frac{W-10}{\sqrt{6}} \leq \frac{8.5-10}{\sqrt{6}}\right) \\
=&P(Z \leq -0.61)
\end{aligned}
$$

Solution: (continued)



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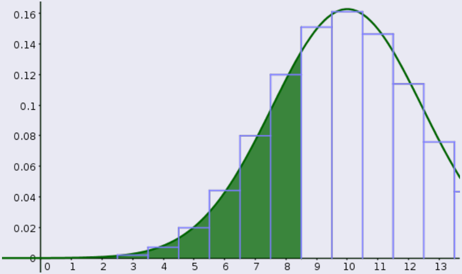
$$
\begin{aligned}
& P(W \leq 8.5) \\
=& P\left(\frac{W-10}{\sqrt{6}} \leq \frac{8.5-10}{\sqrt{6}}\right) \\
=& P(Z \leq -0.61) = 0.2709.
\end{aligned}
$$

from Table 4, Appendix 3.

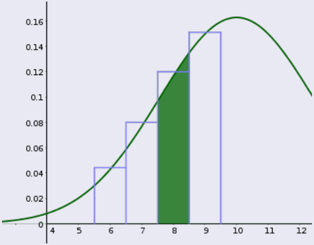
Solution: (continued)



Thus

$$P(W \leq 8.5)$$
$$= P\left(\frac{W-10}{\sqrt{6}} \leq \frac{8.5-10}{\sqrt{6}}\right)$$
$$= P(Z \leq -0.61) = 0.2709.$$

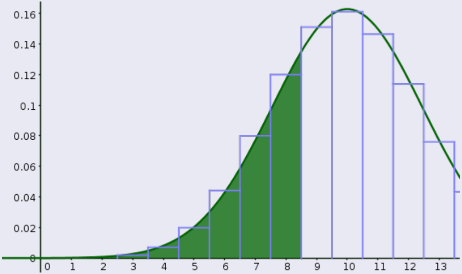
from Table 4, Appendix 3.



Likewise,

$$P(7.5 \leq W \leq 8.5)$$

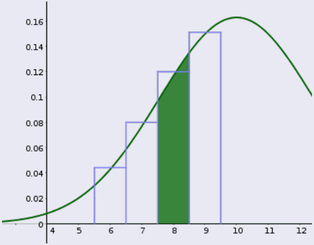
Solution: (continued)



Thus

$$P(W \le 8.5)$$
$$= P\left(\frac{W-10}{\sqrt{6}} \le \frac{8.5-10}{\sqrt{6}}\right)$$
$$= P(Z \le -0.61) = 0.2709.$$

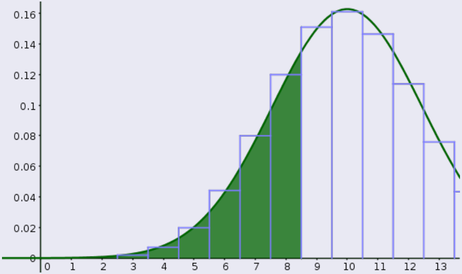
from Table 4, Appendix 3.



Likewise,

$$P(7.5 \le W \le 8.5)$$
$$= P\left(\frac{7.5-10}{\sqrt{6}} \le \frac{W-10}{\sqrt{6}} \le \frac{8.5-10}{\sqrt{6}}\right)$$

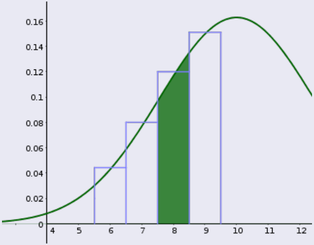
Solution: (continued)



Thus

$$P(W \leq 8.5)$$
$$= P\left(\frac{W-10}{\sqrt{6}} \leq \frac{8.5-10}{\sqrt{6}}\right)$$
$$= P(Z \leq -0.61) = 0.2709.$$

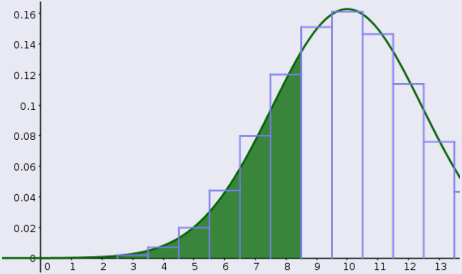
from Table 4, Appendix 3.



Likewise,

$$P(7.5 \leq W \leq 8.5)$$
$$= P\left(\frac{7.5-10}{\sqrt{6}} \leq \frac{W-10}{\sqrt{6}} \leq \frac{8.5-10}{\sqrt{6}}\right)$$
$$= P(-1.02 \leq Z \leq -0.61)$$

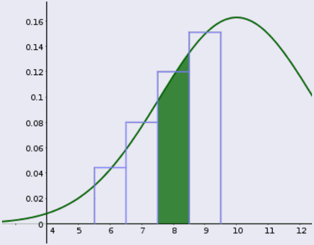
Solution: (continued)



Thus

$$P(W \le 8.5)$$
$$=P\left(\frac{W-10}{\sqrt{6}} \le \frac{8.5-10}{\sqrt{6}}\right)$$
$$=P(Z \le -0.61) = 0.2709.$$

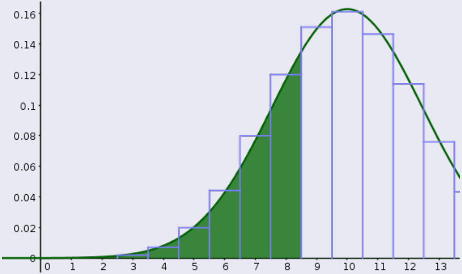
from Table 4, Appendix 3.



Likewise,

$$P(7.5 \le W \le 8.5)$$
$$=P\left(\frac{7.5-10}{\sqrt{6}} \le \frac{W-10}{\sqrt{6}} \le \frac{8.5-10}{\sqrt{6}}\right)$$
$$=P(-1.02 \le Z \le -0.61)$$
$$=0.2709 - 0.1539$$

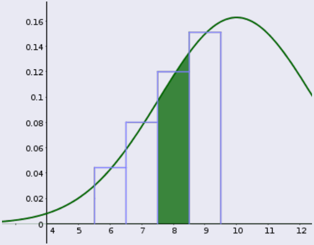
Solution: (continued)



Thus

$$P(W \leq 8.5)$$
$$= P\left(\frac{W-10}{\sqrt{6}} \leq \frac{8.5-10}{\sqrt{6}}\right)$$
$$= P(Z \leq -0.61) = 0.2709.$$

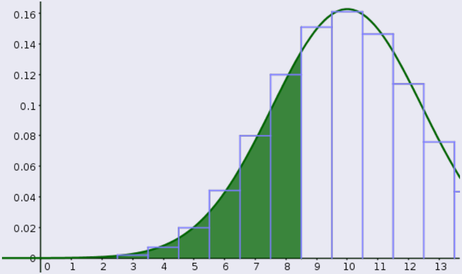
from Table 4, Appendix 3.



Likewise,

$$P(7.5 \leq W \leq 8.5)$$
$$= P\left(\frac{7.5-10}{\sqrt{6}} \leq \frac{W-10}{\sqrt{6}} \leq \frac{8.5-10}{\sqrt{6}}\right)$$
$$= P(-1.02 \leq Z \leq -0.61)$$
$$= 0.2709 - 0.1539 = 0.1170.$$

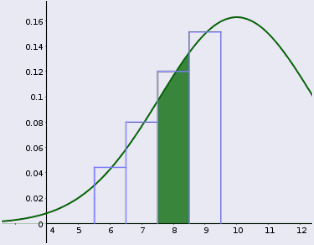
Solution: (continued)



Thus

$$\begin{aligned}
&P(W \leq 8.5) \\
=&P\left(\frac{W-10}{\sqrt{6}} \leq \frac{8.5-10}{\sqrt{6}}\right) \\
=&P(Z \leq -0.61) = 0.2709.
\end{aligned}$$

from Table 4, Appendix 3.



Likewise,

$$\begin{aligned}
&P(7.5 \leq W \leq 8.5) \\
=&P\left(\frac{7.5-10}{\sqrt{6}} \leq \frac{W-10}{\sqrt{6}} \leq \frac{8.5-10}{\sqrt{6}}\right) \\
=&P(-1.02 \leq Z \leq -0.61) \\
=&0.2709 - 0.1539 = 0.1170.
\end{aligned}$$

Note that the approximate values (0.2709 and 0.1170) are very close to the actual values (0.274 and 0.120) calculated earlier.

End of Chapter 7