Math 447 **Final Exam** **Spring 2017**

No books, no notes, only SOA-approved calculators. You must show work, unless the question is a true/false, multiple-choice, or fill-in-the-blank question. Numerical answers should be given to 3 places, e.g. 97.9%.

Name: ______________________________ Section: ____________________

Question	Points	Score
1	6	
2	6	
3	6	
4	6	
5	24	
6	8	
7	17	
8	14	
9	12	
10	18	
11	21	
12	21	
13	26	
Total:	185	

1. (6 points) Suppose we have three boxes:

   - a box containing two gold coins,
   - a box containing two silver coins,
   - a box containing one gold coin and a silver coin.

   We choose a box at random and withdraw one coin at random. It happens to be a gold coin. What is the probability that the other coin in the same box is also a gold coin? Select the option below closest to the correct answer.

   1. ____________

   A. 0.63

   B. 0.56

   C. 0.46

   D. 0.31

2. (6 points) A student answers a multiple-choice examination question that offers four possible answers. Suppose the probability that the student knows the answer to the question is 0.8 and the probability that the student will guess is 0.2. Assume that if the student guesses, the probability of selecting the correct answer is 0.25. If the student correctly answers a question, what is the probability that the student really knew the correct answer?

   2. ____________

3. (6 points) As items come to the end of a production line, an inspector chooses which items are to go through a complete inspection. Ten percent of all items produced are defective. Sixty percent of all defective items go through a complete inspection, and 20% of all good items go through a complete inspection. Given that an item is completely inspected, what is the probability it is defective?

3. ____________

4. (6 points) Recall the original Monty Hall Problem: the game show has three curtains, one of which conceals a car, and the other two of which conceal goats. The contestant chooses a curtain at random, and the host (who knows where the car is) opens another curtain to display a goat. The contestant is offered the opportunity to switch from his original choice to the remaining unopened curtain.

   Suppose now the host does not always open another curtain and offer the option to switch, but chooses each time whether to do so after seeing the contestant's choice of curtain. Assume the host wishes to minimize the contestant's chances of winning the car. Which of the following statements is most accurate?

4. ____________

   A. The contestant should switch curtains whenever offered a choice and his chances of winning are unchanged.
   B. With the correct strategy for offering choices, the host can reduce the contestant's chances of winning the car to less than 1/3.
   C. The contestant should switch curtains whenever offered a choice, but his chances of winning are not the same as in the original problem.
   D. None of the above statements is accurate.

5. Suppose that Y is normally distributed with mean μ and variance σ^2 and that $U = e^Y$.

(a) (6 points) Fill in the blanks to complete the statements about the relationship between the CDFs F_Y and F_U.

$$F_U(u) = P(\underline{\hspace{3cm}}) \qquad \text{by definition of CDF}$$

$$= P(\underline{\hspace{3cm}} \leq u) \qquad \text{by definition of } U$$

$$= P(Y \leq \underline{\hspace{3cm}})$$

$$= \underline{\hspace{3cm}} \qquad \text{by definition of } F_Z$$

(b) (6 points) Find the density function $f_U(u)$. *Hint:* Your answer should depend on μ and σ.

$$f_U(u) = \begin{cases} \underline{\hspace{6cm}} & \text{if } \underline{\hspace{3cm}} \\ \underline{\hspace{6cm}} & \text{if } \underline{\hspace{3cm}} \end{cases}$$

(c) (3 points) Fill in the blank: the definition of the MGF $m_Y(t)$ is $E[\underline{\hspace{3cm}}]$.

(d) (3 points) Use the moment-generating function of Y to find $E[U]$. *Hint:* Your answer should depend on μ and σ.

(d) ______________

(e) (3 points) Use the moment-generating function of Y to find $E[U^2]$. *Hint:* Your answer should depend on μ and σ.

(e) ______________

(f) (3 points) Find $V[U]$. *Hint:* Your answer should depend on μ and σ.

(f) ____________

6. (8 points) Fill in the blanks in the following statement of the central limit theorem. Note that *some variable names have changed* and you will have to adapt the statement accordingly.

   Let $Y_1, Y_2, \ldots, Y_n, \ldots$ be independent and identically distributed random variables with $E[Y_i] = a$ and $V[Y_i] = b^2$. Define

   $$\bar{Y_n} = \underline{\hspace{6cm}} \quad \text{and} \quad U_n = \underline{\hspace{6cm}}$$

   Then the distribution function of U_n converges to the ____________________ normal distribution function as $n \to \infty$. That is,

   $$\lim_{n\to\infty} P(\underline{\hspace{3cm}}) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt$$

   for all u.

7. One-hour carbon monoxide concentrations in air samples from a large city average 12 ppm (parts per million) with standard deviation 9 ppm.

   (a) (2 points) Are carbon monoxide concentrations in air samples from this city approximately normally distributed? Answer "yes" or "no".

   (a) ____________

   (b) (3 points) Why or why not?

(c) (2 points) Suppose we take 100 randomly selected air samples and look at the average carbon monoxide concentration in these samples. Is this average approximately normally distributed? Answer “yes” or “no”.

(c) ____________

(d) (3 points) Why or why not?

(e) (7 points) Find the probability that the average concentration in 100 randomly selected samples will exceed 14 ppm.

(e) ____________

8. Scores on an examination are assumed to be normally distributed with mean 78 and variance 36.

   (a) (6 points) Suppose that students scoring in the top 10% of this distribution are to receive an A grade. What is the minimum score a student must achieve to earn an A grade?

   (a) ____________

(b) (8 points) If it is known that a student's score exceeds 72, what is the probability that his or her score exceeds 84?

(b) ____________

9. Let $Y_1, Y_2, \ldots, Y_5$ be a random sample of size 5 from a normal population with mean 0 and variance 1 and let

$$\bar{Y} = (1/5)\sum_{i=1}^{5} Y_i \qquad \text{and} \qquad W = \sum_{i=1}^{5} Y_i^2$$

(a) (4 points) What is the distribution of $\bar{Y}$ and what are the relevant parameters?

(b) (4 points) What is the distribution of W and what are the relevant parameters?

(c) (4 points) What is the distribution of $\sum_{i=1}^{5}(Y_i - \bar{Y})^2$ and what are the relevant parameters?

10. Consider the following game: A player throws a fair die repeatedly until he rolls a 2, 3, 4, 5, or 6. In other words, the player continues to throw the die as long as he rolls 1s. When he rolls a "non-1," he stops. Let Y be the number of throws needed to obtain the first non-1.

(a) (4 points) Y is a random variable of a type we have studied. What is the name of this type of random variable and what are the relevant parameter(s)?

(b) (3 points) What is the probability that the player tosses the die exactly three times?

(b) ____________

(c) (3 points) What is the expected number of rolls needed to obtain the first non-1?

(c) ____________

(d) (8 points) If he rolls a non-1 on the first throw, the player is paid \$1. Otherwise, the payoff is doubled for each 1 that the player rolls before rolling a non-1. Thus, the player is paid \$2 if he rolls a 1 followed by a non-1; \$4 if he rolls two 1s followed by a non-1; \$8 if he rolls three 1s followed by a non-1; etc. In general, if the player rolls $(Y-1)$ 1s before rolling his first non-1, he is paid 2^{Y-1} dollars. What is the expected amount paid to the player?

(d) ____________

11. Suppose that two continuous random variables Y_1, Y_2 have joint density function given by

$$f(y_1, y_2) = \begin{cases} 6(1 - y_2) & 0 \le y_1 \le y_2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) (3 points) Graph the region of the plane in which f is nonzero.

(b) (6 points) Fill in the blanks to write down a double integral for $P(Y_1 \le 3/4)$. Note that the order of integration is prescribed.

$$\int_{\_\_\_\_\_}^{\_\_\_\_\_} \int_{\_\_\_\_\_}^{\_\_\_\_\_} \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \, dy_2 dy_1$$

(c) (12 points) Find $P(Y_2 \le 1/2 \mid Y_1 \le 3/4)$.

(c) ______________

12. Suppose that Y_1, Y_2, Y_3 are three independent uniform random variables on the interval $[0, a]$. Let $F(y)$ be their common CDF and $f(y)$ their common pdf. Let $Y_{(3)} = \max(Y_1, Y_2, Y_3)$, and G, g be the CDF and PDF of Y_3.

(a) (6 points) Find $G(y)$ in terms of $F(y)$.

(a) ____________

(b) (6 points) Find $g(y)$ in terms of $F(y)$ and $f(y)$, using the previous part.

(b) ____________

(c) (6 points) Suppose that the number of minutes you have to wait for a bus is uniformly distributed on the interval $[0, 15]$. Write an integral for the probability that the longest wait is less than 10 minutes.

(d) (3 points) Find the probability discussed in the previous part.

13. Suppose that X and Y are two independent exponential random variables with mean 1.

(a) (6 points) Graph the region in the (x, y) plane where the joint density function of X and Y is not zero and $y \leq 3x$.

(b) (10 points) Fill in the blanks to write down a double integral for $P(Y \leq 3X)$. Note that the order of integration is prescribed.

$$\int_{\_\_\_\_\_\_}^{\_\_\_\_\_\_} \int_{\_\_\_\_\_\_}^{\_\_\_\_\_\_} \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \, dxdy$$

(c) (10 points) Find $P(Y \leq 3X)$.