No books, no notes, only SOA-approved calculators. You must show work, unless the question is a true/false or fill-in-the-blank question. If you use a your calculator to give a decimal answer, make it accurate to 3 significant figures unless otherwise instructed.

1. If two events, A and B, are such that $P(A) = .5$, $P(B) = .3$, and $P(A \cap B) = .1$, find the following:
   (a) (2 points) $P(A \mid B)$

   Solution:
   $$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{3}$$

   (b) (2 points) $P(A \cup B)$

   Solution:
   $$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.5 + 0.3 - 0.1 = 0.7$$

   (c) (3 points) $P(A \mid A \cup B)$

   Solution:
   $$P(A \mid A \cup B) = \frac{P(A \cap (A \cup B))}{P(A \cup B)} = \frac{P(A)}{P(A \cup B)} = \frac{5}{7}$$

2. Five candidates for a job, who we will call A, B, C, D, and E are seated randomly. What is the probability that A and B are seated next to each other if . . .
   (a) (6 points) the candidates are seated around a circular table?

   Solution: We may assume by rotating the table that A is seated in a fixed position. There are four seats remaining at the table and B is equally likely to be in any of those four seats. Of those seats, two are next to A and two are not, so the answer is $2/4 = \frac{1}{2}$.

   (b) (9 points) the candidates are seated in a row?

   Solution: Consider the position of A: either A is an end of the row, with probability $\frac{2}{5}$, or somewhere in the middle, with probability $\frac{3}{5}$. Again B is equally likely to be in any of the four remaining seats.

   If A is seated at an end of the row, there is only one seat next to A, and so the probability that B is seated next to A is $\frac{1}{4}$. If A is seated in the middle somewhere, there are two seats next to A and the probability that B is seated next to A is $\frac{1}{2}$.

Now by the "Law of Total Probability" the probability that A and B are seated next to each other is

$$\frac{2}{5}\cdot\frac{1}{4}+\frac{3}{5}\cdot\frac{1}{2}=\frac{2}{5}.$$

3. On today's episode of the "Monty Hall" game show, you are shown six cards, all face-down, of which one is the $Q\heartsuit$ and five are black spot cards. The objective is to select the $Q\heartsuit$, which wins \$100.

   The host lets you select a face-down card, and then shows you, from the other five face-down cards, two black spot cards, which he can always do, because he knows where all the cards are. He then offers you the opportunity to switch your choice to one of the three remaining face-down cards.

   We will suppose in what follows that you decide to switch your choice.

   Let G denote the event that your initial selection was the $Q\heartsuit$. Let W denote the event that your new selection (after switching) is the $Q\heartsuit$.

   (a) (3 points) Find $P(\bar{G})$, $P(W \mid \bar{G})$, and $P(W \mid G)$.

   Solution:

   $$P(\bar{G})=\frac{5}{6} \qquad P(W\mid\bar{G})=\frac{1}{3} \qquad P(W\mid G)=0$$

   (b) (4 points) Write an equation for $P(W)$ using the "Law of Total Probability".

   Solution:

   $$P(W)=P(W\mid\bar{G})P(\bar{G})+P(W\mid G)P(G)$$

   (c) (4 points) Find $P(W)$.

   Solution:

   $$P(W)=\frac{1}{3}\cdot\frac{5}{6}+0=\frac{5}{18}$$

4. Males and females are observed to react differently to a given set of circumstances. It has been observed that 70% of the females react positively to these circumstances, whereas only 40% of males react positively. A group of 20 people, 15 female and 5 male, was subjected to these circumstances, and the subjects were asked to describe their reactions on a written questionnaire. A response picked at random from the 20 was negative. What is the probability that it was that of a male?

   Solve this problem according to the following scheme. Make sure to give an appropriate answer for each part.

   (a) (3 points) Establish notation: name the relevant events.

Solution: Let F be the event that a response selected at random is that of a female. (So $\bar{F}$ is the event that a response selected at random is that of a male.)

Let N be the event that a response selected at random is negative. (So $\bar{N}$ is the event that a response selected at random is positive.)

(b) (3 points) In the notation of part (a), what information does the problem give you?

Solution:

$$P(\bar{N} \mid F) = 0.7 \qquad P(\bar{N} \mid \bar{F}) = 0.4 \qquad P(F) = 0.75 \qquad P(\bar{F}) = 0.25$$

(c) (2 points) In the notation of part (a), what probability does the problem ask you to compute?

Solution: $P(\bar{F} \mid N)$

(d) (3 points) Write down the equation that lets you compute this answer, again in the notation of part (a).

Solution: Bayes Formula gives us:

$$P(\bar{F} \mid N) = \frac{P(N \mid \bar{F})P(\bar{F})}{P(N \mid \bar{F})P(\bar{F}) + P(N \mid F)P(F)}$$

(e) (3 points) Solve the problem, giving your answer as a fraction in reduced form or as a decimal number.

Solution: We plug in the numbers from part (b) into the equation from part (d) and get:

$$P(\bar{F} \mid N) = \frac{0.6(0.25)}{0.6(0.25) + (0.3)(0.75)} = 0.4$$

In the above we have used $P(\bar{(A)}) = 1 - P(A)$ without comment.

5. Suppose a random variable Y has probability function p given by

$$p(-1) = \frac{1}{32} \qquad p(0) = \frac{30}{32} \qquad p(1) = \frac{1}{32}$$

(a) (3 points) Find $E[Y]$

Solution: $E[Y] = \frac{1}{32}(-1) + \frac{30}{32}(0) + \frac{1}{32}(1) = 0.$

(b) (3 points) Find $V[Y]$

Solution: $V[Y] = E[Y^2] - 0^2 = \frac{1}{32}(1) + \frac{30}{32}(0) + \frac{1}{32}(1) = \frac{1}{16}.$

(c) (3 points) Fill in the blank in the following statement of Tchebysheff's inequality:

$$P(|Y - \mu| \geq k\sigma) \leq \underline{\frac{1}{k^2}}$$

(d) (2 points) What are μ and σ for the Y of this problem? Give numerical answers.

Solution: $\mu = 0$ and $\sigma = \frac{1}{4}$

(e) (2 points) For the Y of this problem, apply Tchebysheff's inequality to fill in the blank in the equation below:

$$P(|Y - \mu| \geq 1) \leq \underline{\frac{1}{16}}$$

6. A biased coin with probability $p = 0.3$ of landing heads is tossed repeatedly. Let X be the number of the toss on which the first head occurs.

(a) (3 points) What kind of random variable is X, and what are the values of the defining parameter(s)? (Hint: you have only learned about a small number of random variables so far in this course!)

Solution: X is a geometric random variable with parameter $p = 0.3$.

(b) (2 points) Find $P(X = 1)$. (Give a numerical answer.)

Solution: $P(X = 1) = 0.3$

(c) (2 points) Find $P(X = 3)$. (Give a numerical answer.)

Solution: $P(X = 3) = (0.7)^2(0.3) = 0.147$

(d) (4 points) Write an infinite series for the probability that X is an odd number.

Solution: We write this series in terms of $p = 0.3$ and $q = 1 - p = 0.7$:

$$P(X \text{ is odd}) = P(X = 1) + P(X = 3) + P(X = 5) + \cdots = p + q^2p + q^4p + \cdots = \sum_{m=0}^{\infty} pq^{2m}$$

(e) (4 points) Sum your infinite series and find the probability that X is odd.

Solution:

$$\sum_{m=0}^{\infty} pq^{2m} = p\sum_{m=0}^{\infty}(q^2)^m = p\frac{1}{1-q^2} \approx 0.588$$

7. (World Series, revisited) Two teams A and B play a series of games until one team has won 4 games. Assumte that the results of the games are independent, the probability that team A wins any given game is $p = 0.6$, and the probability that team B wins any given game is $q = 1 - p = 0.4$.

(a) (3 points) Find the probability that team A wins exactly 3 of the first 5 games.

Solution: The number of games won by A in the first 5 games is (almost) a binomial random variable $Y \sim \text{Bin}(5, p)$, and the probability is the binomial probability:

$$\binom{5}{3}p^3q^2 = 0.3456$$

We say "almost" above because if one team wins the first four games, the fifth game is not played. We get around this issue by imagining an alterative set of rules under which the fifth game is played, even if one team has already won four games. The probability of A winning 3 of the first 5 games is identical under these rules.

(b) (3 points) Find the probability that team A wins exactly 2 of the first 5 games.

Solution:

$$\binom{5}{2}p^2q^3 = 0.2304$$

(c) (3 points) If the series lasts exactly 6 games, what results are possible for team A in the first 5 games, i.e. how many games of the first 5 can team A possibly win?

Solution: If the series is to last exactly 6 games, one team must win 3 of the first 5 and then win the sixth game. Therefore A must win either 2 or 3 of the first 5 games.

(d) (3 points) What is the probability that the series lasts exactly 6 games?

Solution:

$$P(Y = 2)q + P(Y = 3)p \approx 0.300$$

(e) (3 points) Given that the series lasts exactly 6 games, what is the probability that team A wins the series?

Solution: The binomial coefficients cancel and this conditional probability is

$$\frac{P(Y=3)p}{P(Y=2)q + P(Y=3)p} = \frac{p^3q^2p}{p^2q^3q + p^3q^2p} = \frac{p^2}{q^2+p^2} \approx 0.6923$$

8. Cards are dealt at random and without replacement from a standard 52-card deck. Let Y be the number of kings that appear in the first four cards dealt.

   (a) (2 points) What is the name of the distribution of Y? Hint: we have only learned about a small number of probability distributions, so the answer must be one of these!

   Solution: Y has the hypergeometric distribution.

   (b) (2 points) What are the values of the parameters that define this probability distribution?

   Solution: $N = 52$, $r = 4$, $n = 4$

   (c) (4 points) Find the probability that $Y = 1$.

   Solution: Use the probability function for the hypergeometric random variable, which is in the table of discrete distributions.

   $$P(Y=1) = \frac{\binom{r}{1}\binom{N-r}{n-1}}{\binom{N}{n}} = \frac{\binom{4}{1}\binom{48}{3}}{\binom{52}{4}} \approx 0.256$$

   (d) (2 points) What is the mean of Y?

   Solution: Use the formula for the mean of the hypergeometric random variable, which is in the table of discrete distributions. $E[Y] = nr/N = 16/52 = \frac{4}{13}$.

   (e) (8 points) What is the probability that the second king dealt from the deck is the fifth card dealt? Hint: if the second king dealt is the fifth card dealt, what can we say about the first four cards dealt?

   Solution: Let $K_{2,5}$ be the event that the second king dealt is the fifth card dealt. If this happens, then there must be exactly one king dealt in the first four cards dealt, and the fifth card must be a king. Let K_5 be the event that the fifth card dealt is a king. Thus,

   $$P(K_{2,5}) = P(K_5 \mid Y=1)P(Y=1) = \frac{3}{48}(0.255) \approx 0.0160$$

9. Fill in the blank in the following definition and property of moment generating functions.

(a) (3 points) By definition, the moment generating function of a random variable Y is

$$m_Y(t) = E[\underline{e^{tY}}]$$

10. In the daily production of a certain kind of rope, the number of defects per foot Y is assumed to have a Poisson distribution with mean 2. The profit per foot when the rope is sold is given by X, where $X = 50 - 2Y - Y^2$.

(a) (2 points) Find $E[Y]$

Solution: From the problem statement, $E[Y] = 2$

(b) (4 points) Find $E[Y^2]$

Solution: The Poisson distribution with parameter λ has mean λ and variance λ, so we have $\lambda = 2$ and $V[Y] = 2$. Now $E[Y^2] = V[Y] + E[Y]^2 = 2 + 2^2 = 6$.

(c) (4 points) Find $E[X]$

Solution: Using linearity of expectation,

$$E[X] = E[50 - 2Y - Y^2] = E[50] - 2E[Y] - E[Y^2] = 50 - 2 \cdot 2 - 6 = 40$$