Math 447 **Test 2 Solutions** **Fall 2019**

No books, no notes, only SOA-approved calculators. You must show work, unless the question is a true/false or fill-in-the-blank question, or you are told elsewhere in the statement of the problem that no work is necessary. You may give numerical answers either as reduced fractions, or as decimals accurate to 3 significant figures unless otherwise instructed.

Section Number	Instructor	Meeting Time
1	Dai	8:00
2	Karagueuzian	11:20
3	Hanson	8:00

1. (10 points) (Bertrand Box Puzzle) We have five identical-looking boxes, each of which contains two coins. Of the five boxes,

   - two boxes contain two silver coins each,
   - two boxes contain two gold coins each,
   - one box contains a silver coin and a gold coin.

   A box is selected at random and a coin drawn from the box. This coin turns out to be gold. What is the probability that the other coin in the selected box is also gold?

   Solution: There are 10 equally likely possible outcomes of selecting a box and then a coin. Of these 10 outcomes, 5 of them result in the selection of a gold coin. Of these 5 cases in which a gold coin is selected, 4 of them are cases in which the other coin in the box is gold.

   So the answer is 4/5.

2. Two methods, A and B, are available for teaching a certain industrial skill. The failure rate is 20% for A and 10% for B. However, B is more expensive and hence is used only 30% of the time. (A is used the other 70%.)

   (a) (8 points) A worker was taught the skill by one of the methods but failed to learn it correctly. What is the probability that she was taught by method A?

   Solution: Let A denote the event that a worker was taught using method A. Let B denote the event that a worker was taught using method B. Let F be the event that the worker failed to learn the skill.

   The problem gives us the information: $P(A) = 0.7$, $P(B) = 0.3$, $P(F \mid B) = 0.1$, and $P(F \mid A) = 0.2$.

   We are asked to compute $P(A \mid F)$, and we can apply Bayes' Formula.

$$P(A \mid F) = \frac{P(F \mid A)P(A)}{P(F \mid A)P(A) + P(F \mid B)P(B)} = \frac{0.2(0.7)}{0.2(0.7) + 0.1(0.3)} = \frac{14}{17} \approx 0.8235$$

(b) (8 points) What is the probability that a worker, chosen at random from those who were taught the skill, failed to learn the skill?

> **Solution:** Using the notation above, we apply the law of total probability.
>
> $$P(F) = P(F \mid A)P(A) + P(F \mid B)P(B) = 0.17$$
>
> Note that $P(F)$ is the denominator in part (a).

3. (For this question you may wish to refer to the table of the normal distribution attached to the back of your test.) We take as our model of the grade point averages (GPAs) of students at a certain college a normal distribution with mean 2.4 and standard deviation 0.8. The college has exactly 10,000 students.

(a) (3 points) If students possessing a GPA less than 1.9 are placed on probation, what percentage of the students will be on probation?

> **Solution:** The z-score corresponding to 1.9 is $(1.9 - 2.4)/0.8 = -0.625$. We find in the table that the fraction of the standard normal distribution Z above 0.625 is $P(Z > 0.625) \approx (0.2676 + 0.2643)/2 = 0.26595$. Thus, by symmetry $P(Z < -0.62595) = 0.26595$, so approximately 26.6% of students will be on probation.

(b) (5 points) If the college administration wishes to put students with the lowest 4% of GPAs on probation, what should the cutoff GPA for probation be?

> **Solution:** From the table, $P(Z > 1.75) = 0.0401$. By the symmetry of the normal distribution, $P(Z < -1.75) = 0.0401$. So the z-score -1.75 is the cutoff for the lowest 4.01% of GPAs. This corresponds to a GPA of $2.4 - 1.75 \cdot 0.8 = 1.0$. So the cutoff GPA should be about 1.0, or perhaps a little less to account for the difference between 4.01% and 4%.

(c) (5 points) How many students, *according to the model,* have GPAs which are *less than zero?*

> **Solution:** From the table $P(Z > 3) = 0.00135$. By symmetry, $P(Z < -3) = 0.00135$. The problem tells us that there are $10,000$ students, so according to the model there are $(10,000) \cdot 0.00135 = 13.5$ students with GPAs less than zero. Students are usually thought of as discrete rather than continuous, so *according to the model, 13 or 14 students have GPAs less than zero.* Our university administration thinks in terms of "FTEs", or full-time-equivalent students, so arguably there is such a thing as a fractional student. Maybe the answer 13.5 is okay?

(d) (3 points) In the real world, how many students will actually have a GPA which is *less than zero?*

> **Solution:** On the standard A to F scale, an F is 0.0 and there is no way to get a negative GPA, so *no students will actually have a GPA which is less than zero.*

(e) (3 points) Should your answers to (c) and (d) be the same? Why or why not?

Solution: No, because the model is only an approximation to the real world, and does not perfectly correspond to it.

4. Suppose that Y_1 and Y_2 are discrete random variables whose joint probability function $p(y_1, y_2)$ is given by the following table:

y_2 \ y_1	0	1
0	2/5	1/15
1	2/15	1/15
2	4/15	1/15

(a) (4 points) Find the marginal probability function $p_1(y_1)$ of Y_1.

Solution: Total the columns: $p_1(0) = 4/5$, $p_1(1) = 1/5$ and $p_1(\text{other}) = 0$.

(b) (6 points) Find the conditional probability function $p(y_2 \mid y_1)$ of Y_2 given Y_1. Put your answer in a table, similar to the table above.

Solution: Use the formula

$$p(y_2 \mid y_1) = \frac{p(y_1, y_2)}{p_1(y_1)}$$

and compute to get the table:

$p(y_2 \mid y_1)$ y_2 \ y_1	0	1
0	1/2	1/3
1	1/6	1/3
2	1/3	1/3

(c) (2 points) Are Y_1 and Y_2 independent? (Yes or no; no work required.)

Solution: No, because the joint probability function is not the product of the marginal probability functions. (Or because the conditional probability function is not independent of y_1.)

5. Suppose Y_1 and Y_2 are continuous random variables with joint density function given by

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & \text{for } y_1 > 0 \text{ and } y_2 > 0 \\ 0, & \text{otherwise} \end{cases}$$

(a) (2 points) Are Y_1 and Y_2 independent? (Yes or no; no work required.)

Solution: Yes, because the given probability density function is the product of a function only of y_1, namely e^{-y_1}, and a function only of y_2, namely e^{-y_2}. (See Theorem 5.5 in the textbook.) This is actually the product of two exponential probability density functions, each with parameter $\beta = 1$, but we don't need to know this for part(a).

(b) (3 points) The marginal distribution of Y_1 is one of the continuous distributions studied in Chapter 4 and listed in the table attached to the end of this test. What is the name of this distribution and what are the value(s) of the defining parameters?

Solution: The marginal distribution of Y_1 (and also of Y_2) is exponential with parameter $\beta = 1$. This is the same as gamma, with parameters $\alpha = 1$ and $\beta = 1$.

(c) (4 points) Find $E[Y_1 - Y_2]$. Use the back of the previous page if necessary.

Solution: $E[Y_1 - Y_2] = E[Y_1] - E[Y_2] = 1 - 1 = 0$, where we have taken the expectation of the exponential distribution from the table of continuous distributions. Alternatively, note that the marginal densities are the same, so $E[Y_1] = E[Y_2]$.

(d) (6 points) Find $V[Y_1 - Y_2]$. Use the back of the previous page if necessary.

Solution: Note that Y_1 and $(-1)Y_2$ are independent, so that

$$V[Y_1 + (-1)Y_2] = V[Y_1] + V[(-1)Y_2] = V[Y_1] + (-1)^2 V[Y_2] = 1 + 1 = 2.$$

A common mistake: $V[Y_1 - Y_2] = V[Y_1] - V[Y_2] = 0$. The first step is false, and the conclusion that the variance is zero is wildly implausible. In this course, a random variable has variance zero if and only if it is constant. (We can't explain yet what "almost surely" constant means.)

6. Suppose Y is a continuous random variable with density function $f(y)$ and distribution function $F(y)$.

(a) (3 points) Express $P(2 < Y < 5)$ as an integral.

Solution:

$$\int_2^5 f(y)\,dy$$

(b) (3 points) Express $P(1 < Y < 4)$ using the distribution function.

Solution: $F(4) - F(1)$

(c) (3 points) Find the derivative $\frac{d}{dy}F(y)$

Solution: $\frac{d}{dy}F(y) = f(y)$

7. Let Y_1 and Y_2 be two uncorrelated random variables with $V[Y_1] = a$, $V[Y_2] = b$. Define two new random variables $U_1 = Y_1 + Y_2$ and $U_2 = Y_1 - Y_2$.

   (a) (3 points) Find $\text{Cov}(U_1, U_2)$ in terms of a and b.

   Solution: Use the bilinearity of covariance:

   $$\text{Cov}(U_1, U_2) = \text{Cov}(Y_1 + Y_2, Y_1 - Y_2) = \text{Cov}(Y_1, Y_1 - Y_2) + \text{Cov}(Y_2, Y_1 - Y_2) =$$
   $$\text{Cov}(Y_1, Y_1) + \text{Cov}(Y_1, -Y_2) + \text{Cov}(Y_2, Y_1) + -\text{Cov}(Y_2, Y_2) = a - b$$

   (b) (3 points) Find the correlation coefficient of U_1 and U_2 in terms of a and b.

   Solution:
   $$\rho = \frac{\text{Cov}(U_1, U_2)}{\sqrt{V[U_1]V[U_2]}} = \frac{a - b}{\sqrt{(a + b)(a + b)}} = \frac{a - b}{a + b}$$

   (c) (4 points) For what values of a and b is this correlation coefficient zero?

   Solution: This correlation coefficient is zero if $a = b$.

8. Suppose Y is a continuous random variable with probability density function given by

   $$f(y) = \begin{cases} 4y^2 e^{-2y}, & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

   (a) (3 points) What kind of random variable is Y and what are the values of the defining parameters?

   Solution: Y is a gamma random variable with parameters $\alpha = 3$ and $\beta = 0.5$.

   (b) (4 points) Find $E[Y]$ and $V[Y]$.

   Solution: Look it up in the table: $E[Y] = 1.5$ and $V[Y] = 0.75$.

   (c) (5 points) Find $E[\sqrt{Y}]$.

Solution: Write down the integral for the expectation and apply the gamma integral formula:

$$E[\sqrt{Y}] = \int_0^\infty y^{1/2} 4y^2 e^{-2y}\, dy = 4\int_0^\infty y^{a-1} e^{-y/b}\, dy = 4b^a\Gamma(a),$$

where $a = 7/2$ and $b = 1/2$. We can simplify using $\Gamma(a) = (a-1)\Gamma(a-1)$ and $\Gamma(1/2) = \sqrt{\pi}$ to get

$$4\left(\frac{1}{2}\right)^{7/2}\Gamma\left(\frac{7}{2}\right) = 4\left(\frac{1}{2}\right)^{3}\left(\frac{1}{2}\right)^{1/2}\left(\frac{7}{2}-1\right)\Gamma\left(\frac{7}{2}-1\right) =$$

$$\frac{1}{2\sqrt{2}}\frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5}{4\sqrt{2}}\left(\frac{5}{2}-1\right)\Gamma\left(\frac{5}{2}-1\right) = \frac{15}{8\sqrt{2}}\left(\frac{3}{2}-1\right)\Gamma\left(\frac{3}{2}-1\right) = \frac{15\sqrt{\pi}}{16\sqrt{2}}$$

9. (10 points) I work in midtown and have two favorite restaurants, one uptown and one downtown. I leave work every day at a random time uniformly distributed between 5 and 6 pm. I go to the subway and take the uptown or downtown train, whichever arrives first, to one of the restaurants.

The uptown and downtown trains arrive at the station with exactly the same frequency. But over a long period of time I find that I go to the uptown restaurant on about 90% of days, and the downtown restaurant on about 10% of days.

How is this possible? (Hint: a correct answer will involve writing down a timetable for the trains.)

Solution: Here is one possible timetable for the arrival times of the trains at our station that results in going uptown on 90% of the days:

Uptown	4:59	5:09	5:19	5:29	5:39	5:49	5:59	6:09
Downtown	5:00	5:10	5:20	5:30	5:40	5:50	6:00	6:10

The uptown and downtown trains both arrive every 10 minutes, that is, with the same frequency.

Notice that if I arrive at the station between 5:00 and 5:09, 5:10 and 5:19, 5:20 and 5:29, 5:30 and 5:39, 5:40 and 5:49, or 5:50 and 5:59, I will take the uptown train, and that this is a total of $54 = 9 \cdot 6$ minutes. If I arrive at a time not in the list of the previous sentence, I will take the downtown train. Thus the probability of taking the uptown train is $54/60 = 90\%$.

Note that other timetables are possible; the trains need not come every 10 minutes. The easiest way is that the trains both come an integer number of times per hours and that the uptown train comes $9/10$ of the way through periods between the arrivals of the downtown train, as in the example timetable above.