Math 447 **Test 1 Solutions** **Fall 2017**

No books, no notes, only SOA-approved calculators. You must show work, unless the question is a true/false or fill-in-the-blank question. Numerical answers should be given to 3 places, e.g. 97.9%.

1. (6 points) Three prisoners, A, B and C, are in separate cells serving life sentences. The governor has selected two of them at random to be pardoned, but the identities of the two are kept secret. Prisoner A considers asking a friendly guard to tell him the name (B or C) of a prisoner other than himself that will be released, but hesitates based on the following rationale: at the prisoner's present state of knowledge, the probability of being released is 2/3, but after he knows the answer, the probability of being released will become 1/2, since there will be two prisoners (including himself) whose fate is unknown and exactly one of the two will be released.

   Which of the following statements best describes the prisoner's reasoning and conclusion about the probability (p) of release after inquiring with the guard?

   A. The conclusion $p = 1/2$ is correct, but the reasoning is incorrect.

   B. The conclusion $p = 1/2$ is incorrect, and the reasoning is also incorrect.

   C. The conclusion $p = 1/2$ is correct, and the reasoning is correct also.

   D. The reasoning is partly right, and a corrected version of it shows that $p = 1/3$.

1. ____________

Solution: This is the Monty Hall problem in disguise. The prisoner's probability of release is unchanged by his acquisition of the information. This should be clear, since neither the prisoner nor the guard have anything to do with the decision, which was made in advance of their conversation anyway. A case-by-case analysis as in the original problem will produce the answer 2/3, and you may wish to try this on your own.

2. (6 points) Many public schools are implementing a "no-pass, no-play" rule for athletes. Under this system, a student who fails a course is disqualified from participating in extracurricular activities during the next grading period. Suppose that the probability is 0.15 that an athlete who has not previously been disqualified will be disqualified next term. For athletes who have been previously disqualified, the probability of disqualification next term is 0.5. If 30% of the athletes have been disqualified in previous terms, which of the following numbers is closest to the probability that a randomly selected athlete will be disqualified during the next grading period?

   A. 0.15

   B. 0.20

   C. 0.25

   D. 0.30

2. ____________

Solution: This is problem 2.174.

3. (6 points) A bag contains 99 normal (fair) coins and one gaffed coin which has two heads. A single coin is selected at random from the bag and flipped 6 times. It comes up heads every time. The chance that it is actually the two-headed coin is closest to:

   A. 0.23

   B. 0.43

   C. 0.63

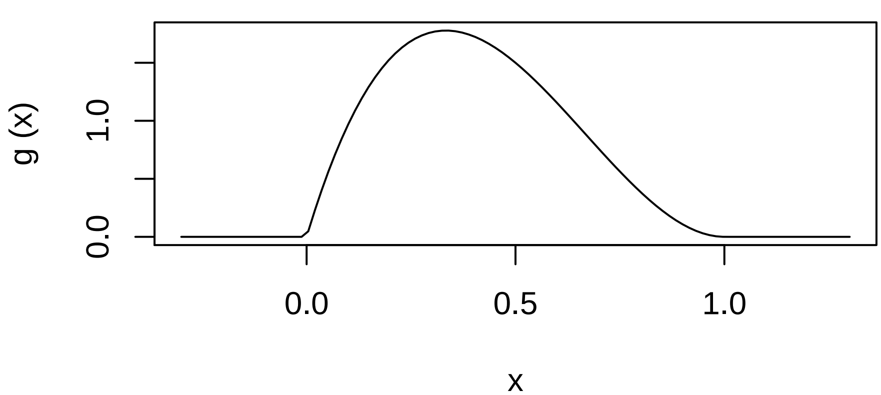
   D. 0.83

3. ____________

Solution: Apply Bayes' Rule. Write G for the event that the coin is gaffed and H for the event that the coin comes up heads 6 times in a row. Then we have $P(G) = \frac{1}{100}$, $P(H \mid \bar{G}) = \frac{1}{64}$, and $P(H \mid G) = 1$.

$$P(G \mid H) = \frac{P(H \mid G)P(G)}{P(H \mid G)P(G) + P(H \mid \bar{G})P(\bar{G})} = \frac{1 \cdot \frac{1}{100}}{1 \cdot \frac{1}{100} + \frac{1}{64} \cdot \frac{99}{100}} \approx 0.392$$

4. (6 points) Consider the following graph of a probability density function:



From which of the following distributions does the probability density function graphed above arise?

A. the beta distribution with parameters $\alpha = 3$ and $\beta = 3$

B. the beta distribution with parameters $\alpha = 5$ and $\beta = 5$

C. the normal distribution with parameters $\mu = 0.3$ and $\sigma = 0.5$

D. the beta distribution with parameters $\alpha = 2$ **and** $\beta = 3$

4. ____________

Solution: The graph is not symmetric, but the graphs corresponding to the the first three distributions are. Also, you can compute the mean and standard deviation of the distributions to check for a match.

5. Suppose that two continuous random variables Y_1, Y_2 have joint density function given by

$$f(y_1, y_2) = \begin{cases} e^{-y_1} & 0 \le y_2 \le y_1 \le \infty \\ 0 & \text{otherwise} \end{cases}$$

(a) (6 points) Find the marginal density function of Y_1. Hint: you may need to include cases, as in the function definition above.

Solution: The marginal density of Y_1 is

$$f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_2 = \begin{cases} \int_0^{y_1} e^{-y_1} dy_2 & 0 \le y_1 < \infty \\ \int_{-\infty}^{\infty} 0 dy_2 & y_1 < 0 \end{cases}$$

This can be simplified to:

$$f_1(y_1) = \begin{cases} y_2 e^{-y_1}|_0^{y_1} & 0 \le y_1 < \infty \\ 0 & y_1 < 0 \end{cases} = \begin{cases} y_1 e^{-y_1} & 0 \le y_1 < \infty \\ 0 & y_1 < 0 \end{cases}$$

Note that if your conditions depends on y_2, you have the wrong answer. The marginal density of Y_1 does not depend on y_2.

(b) (6 points) Find the conditional density function of Y_2 given Y_1. Hint: you may need to include cases, as in the function definition above.

Solution: The conditional density of Y_2 given Y_1 is

$$f(y_2 \mid y_1) = \frac{f(y_1, y_2)}{f_1(y_1)} = \begin{cases} e^{-y_1}/y_1 e^{-y_1} & 0 \le y_2 \le y_1 < \infty \\ 0 & 0 \le y_1 \le y_2 < \infty \\ \text{undefined} & y_1 < 0 \end{cases}$$

This can be simplified to:

$$f(y_2 \mid y_1) = \begin{cases} 1/y_1 & 0 \le y_2 \le y_1 < \infty \\ 0 & 0 \le y_1 \le y_2 < \infty \\ \text{undefined} & y_1 < 0 \end{cases}$$

Note that if you do not have all three cases, then your answer is wrong.

(c) (3 points) Are Y_1 and Y_2 independent? (No reasoning required.)

(c) ____________

Solution: No, by Theorem 5.5.

Solution: This is based on problem 5.33.

6. Suppose that two continuous random variables Y_1, Y_2 are such that the conditional density function of Y_2 given Y_1 is defined for $y_1 \in (0, 1)$ and there we have:

$$f(y_2 \mid y_1) = \begin{cases} 1/y_1 & 0 \le y_2 \le y_1 \\ 0 & \text{otherwise,} \end{cases}$$

and that the marginal density of Y_1 is uniform on the interval $[0, 1]$.

(a) (4 points) The conditional distribution of Y_2 given $Y_1 = y_1$ is one of the six types of continuous distributions we have studied. What is the name of this distribution and what is(are) the relevant parameter(s)?

Solution: The conditional distribution of Y_2 given $Y_1 = y_1$ is *uniform* on the interval $[0, y_1]$. Alternatively, using the parameters in the text, you could say $\theta_1 = 0$ and $\theta_2 = y_1$.

(b) (6 points) Find the joint density function of Y_1 and Y_2. Hint: you may need to include cases, as in the function definition above.

Solution: By definition we have

$$f(y_2 \mid y_1) = \frac{f(y_1, y_2)}{f_1(y_1)} \quad \text{so that} \quad f(y_1, y_2) = f_1(y_1) f(y_2 \mid y_1).$$

Note that $f_1(y_1)$ must be zero where $f(y_2 \mid y_1)$ is undefined. Thus we have

$$f(y_2 \mid y_1) = \begin{cases} 1/y_1 & 0 \le y_2 \le y_1 < 1 \\ 0 & y_1 \in (0,1) \text{ and } y_2 \notin (0, y_1) \\ \text{undefined} & y_1 \notin (0,1) \end{cases} \cdot \begin{cases} 1 & 0 \le y_1 \le 1 \\ 0 & \text{otherwise,} \end{cases}$$

and we can rewrite this as

$$f(y_2 \mid y_1) = \begin{cases} 1/y_1 & 0 \le y_2 \le y_1 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) (8 points) Write down a double integral representing $E[Y_2]$. Be clear about the limits and order of integration.

Solution:

$$E[Y_2] = \int_0^1 \int_0^{y_1} \frac{y_2}{y_1}\, dy_2 dy_1$$

(d) (8 points) Find $E[Y_2]$.

Solution: The answer is $\frac{1}{4}$. You can either do the integral in (c) or apply the law of iterated expectation: $E[Y_2] = E[E[Y_2 \mid Y_1]]$. Since $Y_2 \mid Y_1$ is uniform on $[0, y_1]$ the expectation is $Y_1/2$, and since Y_1 is uniform on $[0, 1]$ the expectation of $Y_1/2$ is $(1/2)/2 = \frac{1}{4}$.

(e) (8 points) Find $V[Y_2]$.

Solution: The answer is $\frac{7}{144}$. You can either compute $E[Y_2^2] - E[Y_2]^2$ using the result of of part (d) and integration for

$$E[Y_2^2] = \int_0^1 \int_0^{y_1} \frac{y_2^2}{y_1}\, dy_2 dy_1$$

or apply the result of theorem 5.15:

$$V[Y_2] = V[E[Y_2 \mid Y_1]] + E[V[Y_2 \mid Y_1]]$$

These last two terms can be computed using the fact that $Y_2 \mid Y_1$ is uniform on $[0, y_1]$ and Y_1 is uniform on $[0, 1]$.

Solution: This is based on problem 5.34.

7. Suppose that W is normal with mean 3 and variance 1. Let $Y = 1 - 2W$.

(a) (2 points) What is the mean of Y?

(a) ____________

(b) (2 points) What is the variance of Y?

(b) ____________

(c) (3 points) True or false: Y is normally distributed. A. True B. False

(c) ____________

(d) (5 points) Give a reason for your answer to the previous part.

Solution: Use the linearity of expectation and the properties of variance to get $E[Y] = -5$ and $V[Y] = 4$. Y is normally distributed, because we can compute the moment generating function of Y and see that it is the same of the mgf of the normal random variable with mean -5 and variance 4:

$$m_Y(t) = E[e^{t(1-2W)}] = e^t E[e^{(-2t)W}] = e^t m_W(-2t) = e^t e^{3(-2t)+\frac{1^2(-2t)^2}{2}} = e^{-5t+\frac{2^2t^2}{2}}.$$

8. Suppose X has the gamma distribution with parameters $\alpha = 2$ and $\beta = 1$. Let $Z = 3X - 2$.

(a) (3 points) True or false: Z has the gamma distribution. A. True B. False

(a) ____________

(b) (5 points) Give a reason for your answer to the previous part.

Solution: Z does not have the gamma distribution, because a gamma-distributed random variable takes only nonnegative values, while Z can be negative.

9. The SAT and ACT college entrance exams are taken by thousands of students each year. The mathematics portions of each of these exams produce scores that are approximately normally distributed. In recent years, SAT mathematics exam scores have averaged 480 with standard deviation 100. The average and standard deviation for ACT mathematics scores are 18 and 6, respectively.

(a) (5 points) An engineering school sets 550 as the minimum SAT math score for new students. What percentage of students will score below 550 in a typical year?

(a) ____________

(b) (7 points) What score should the engineering school set as a comparable standard on the ACT math test?

(b) ____________

Solution: This is problem 4.77.

10. The random variables Y_1 and Y_2 are such that $E[Y_1] = 4$, $E[Y_2] = -1$, $V[Y_1] = 2$, and $V[Y_2] = 4$. (For this problem only answers are necessary, no reasoning or explanation is required.)

(a) (3 points) What is $\text{Cov}(Y_1, Y_1)$?

(a) ______________

(b) (4 points) Is it possible that $\text{Cov}(Y_1, Y_2) = 7$? Your answer should be "yes" or "no". (Hint: if the covariance has this value, what is the correlation coefficient?)

(b) ______________

(c) (4 points) What is the maximum possible value of $\text{Cov}(Y_1, Y_2)$?

(c) ______________

Solution: This is problem 5.97.