Let X be a set, $\{A_i : i \in I\}$ a collection of subsets of X, $B \subseteq X$. We say that the sets $A_i, i \in I$ cover B, if $B \subseteq \bigcup_{i\in I} A_i$. In particular, we say that $\{A_i : i \in I\}$ is a covering of X if $X = \bigcup_{i \in I} A_i$.

Definition: A topological space X is called compact if any covering $\{U_i : i \in I\}$ of X by open sets has a finite subset which still covers X, i.e. has a finite subcovering. In other words, there is a finite set of indices $i_1, \dots, i_k \in I$ such that $X = U_{i_1} \cup \dots \cup U_{i_k}$.

Example: (1) Every finite topological space is compact.

(2) The subspace $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ of $\mathbb{R}$ is compact. In fact, let $\{U_i : i \in I\}$ be an open covering of X. Then $0 \in U_j$ for some $j \in I$. Since $1, \frac{1}{2}, \frac{1}{3}, \dots$ converges to 0, U_j contains all but a finite number of $1, \frac{1}{2}, \frac{1}{3}, \dots$, i.e. $\frac{1}{k} \in U_j$ for all $k > N$ for some N. Now $1 \in U_{i_1}, \frac{1}{2} \in U_{i_2}, \dots, \frac{1}{N} \in U_{i_N}$ for some $i_1, \dots, i_N \in I$. Thus the sets $U_{i_1}, \dots, U_{i_N}, U_j$ cover X. ☒

Exercise: If $a_1, a_2, \dots$ is a sequence in a topological space X which converges to $a \in X$ then the subset $\{a, a_1, a_2, \dots\}$ is compact.

Non-example: (1) $\mathbb{R}$ is not compact. The covering $\{(-n, n) : n \in \mathbb{N}\}$ has no finite sub-coverings.

(2) Any infinite discrete space is not compact.

(3) (X, d) metric space s.t. X is not bounded (i.e. there are points in X at arbitrary large distance) then X is not compact.

Exercise: Prove (3) in details.

Theorem (Heine–Borel). The interval $[0,1]$ is compact.

Proof: Consider an open cover $\{U_i : i \in I\}$ of $[0,1]$. Let
$T = \{t \in [0,1] : [0,t]$ can be covered by finitely many sets from $\{U_i : i \in I\}\}$.
If $T \ni 1$, then we are done. Assume that $1 \notin T$. We will show that this leads to a contradiction by proving that both T and $[0,1] - T$ are open (which contradicts the connecteness of $[0,1]$).

Note that: ① $0 \in T$; ② if $a \in T$ then $[0,a] \subseteq T$. $\triangle$

Let $a \in T$. Then $a < 1$ and $a \in U_j$ for some $j \in I$. Since U_j is open, there is $\varepsilon > 0$ such that $[a, a+\varepsilon) \subseteq U_j$. Now $[0,a]$ is covered by finitely many U_i's and $[a, a+\frac{\varepsilon}{2}] \subseteq U_j$, i.e. $[a, a+\frac{\varepsilon}{2}]$ is covered by U_j. It follows that $[0, a+\frac{\varepsilon}{2}] = [0,a] \cup [a, a+\frac{\varepsilon}{2}]$ is also covered by finitely many of the U_i's. Thus $a+\frac{\varepsilon}{2} \in T$ and $[0, a+\frac{\varepsilon}{2}) \subseteq T$. We showed that with every point $a \in T$, an open neighborhood $[0, a+\frac{\varepsilon}{2})$ of a is contained in T. Thus T is open.

Now suppose $b \notin T$. Then $b > 0$ and $b \in U_k$ for some $k \in I$. Since U_k is open, we have $(b-\varepsilon, b] \subseteq U_k$ for some $\varepsilon > 0$. If there was $t \in T$ such that $t \in (b-\varepsilon, b]$, then $[0,t]$ would be covered by finitely many U_i's and $[t,b] \subseteq U_k$, so $[0,b] = [0,t] \cup [t,b]$ would also be covered by finitely many U_i's, which is false. It follows that $(b-\varepsilon, b] \cap T = \emptyset$, hence $(b-\varepsilon, 1] \cap T = \emptyset$ (by $\triangle$). Thus we proved that $[0,1] - T$ is open. Since T, $[0,1]-T$ are non-empty, disjoint, open sets s.t. $[0,1] = T \cup ([0,1] - T)$, $[0,1]$ is not connected, a contradiction. AS.

Proposition: Let $\mathcal{B}$ be a basis of a topological space X. Then X is compact iff any open cover of X by sets from $\mathcal{B}$ has a finite subcover.

Pf: $\Rightarrow$ clear

$\Leftarrow$ Let $\{U_i : i \in I\}$ be an open cover of X. For any $x \in I$ there is $i(x) \in I$ s.t. $x \in U_{i(x)}$. Since $\mathcal{B}$ is a basis, there is $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U_{i(x)}$. The set $\{B_x : x \in X\}$ is a cover of X by open sets from $\mathcal{B}$. Thus there is a finite subcover: there are $x_1, \dots, x_m \in X$ s.t. $B_{x_1}, \dots, B_{x_m}$ cover X. Since $B_{x_i} \subseteq U_{i(x_i)}$ for all i, we conclude that $U_{i(x_1)}, U_{i(x_2)}, \dots, U_{i(x_m)}$ is a finite covering of X. $\boxtimes$

Theorem: If X, Y are compact then $X \times Y$ is compact.

Proof: Consider an open cover $\{U_i \times V_i : i \in I\}$ of $X \times Y$ by open basic sets (so U_i is open in X, V_i is open in Y).
Fix $y \in Y$. Given $x \in X$, the point $(x, y) \in U_{i_y(x)} \times V_{i_y(x)}$ for some $i_y(x) \in I$. The collection $\{U_{i_y(x)} : x \in X\}$ is an open cover of X. Since X is compact, $U_{i_y}(x_1), \dots, U_{i_y}(x_m)\}$ is a finite subcover of X for some $x_1, \dots, x_m \in X$. Let $V_y = V_{i_y(x_1)} \cap \dots \cap V_{i_y(x_m)}$. Then V_y is open in Y and $y \in V_y$. Also the "strip" $X \times V_y$ is covered by finitely many of the sets $\{U_i \times V_i : i \in I\}$:

$$X \times V_y = U_{i_y(x_1)} \times V_y \cup \dots \cup U_{i_y(x_m)} \times V_y \subseteq U_{i_y(x_1)} \times V_{i_y(x_1)} \cup \dots \cup U_{i_y(x_m)} \times V_{i_y(x_m)}.$$

Now the set $\{V_y : y \in Y\}$ is ~~a finite cover~~ an open cover of Y, so it has a finite subcover: $Y = V_{y_1} \cup \dots \cup V_{y_n}$ for some $y_1, y_2, \dots, y_n \in Y$. This means that $X \times Y = X \times V_{y_1} \cup \dots \cup X \times V_{y_n}$.

We see that $X\times Y$ is a union of finitely many "strips" $X\times V_{y_1}\cup\dots\cup X\times V_{y_n}$, and each strip is covered by finitely many of the sets $\{U_i\times V_i : i\in I\}$. It follows that $X\times Y$ is also covered by finitely many of the sets $\{U_i\times V_i : i\in I\}$. $\boxed{\times}$

It is more difficult to prove the following

<u>Theorem (Tichonov)</u>: Product of any collection of compact topological spaces is compact.

The following useful result expresses compactness in terms of closed sets.

<u>Theorem</u>: A topological space X is compact iff for any collection $\{A_i : i\in I\}$ of <u>closed</u> subsets of X such that $\bigcap_{i\in I} A_i=\phi$, the is a finite subcollection $A_{i_1},\dots,A_{i_k}$ such that $A_{i_1}\cap\dots\cap A_{i_k}=\phi$.

<u>Proof</u>: Note that: ① A_i is closed iff $X-A_i$ is open

② $\bigcap_{i\in I} A_i=\phi$ if and only if $\bigcup_{i\in I} X-A_i=X$

(since $X-\bigcap_{i\in I} A_i=\bigcup_{i\in I}(X-A_i)$)

$\Longrightarrow$ Let X be compact. If $\{A_i : i\in I\}$ consists of closed sets s.t. $\bigcap_{i\in I} A_i=\phi$, then $\{X-A_i : i\in I\}$ is an open cover of X. Hence $(X-A_{i_1})\cup\dots\cup(X-A_{i_m})=X$ for some ~~finite~~ $i_1,\dots,i_m\in I$ which is the same as $A_{i_1}\cap\dots\cap A_{i_m}=\phi$.

$\Longleftarrow$ Let $\{U_i : i\in I\}$ be an open cover of X. Then $X-U_i$ are closed and $\underbrace{\bigcap_{i\in I}(X-U_i)}_{}=X-\left(\bigcup_{i\in I} U_i\right)=\phi$. Thus there are

$i_1, \dots, i_m \in I$ such that $(X - U_{i_1}) \cap \dots \cap (X - U_{i_m}) = \phi$. This is the same as $U_{i_1} \cup \dots \cup U_{i_m} = X$, so we get a finite subcover. $\boxtimes$

<u>Theorem:</u> Let A be a closed subset of a compact space X. Then A is compact.

<u>Pf</u>: Let $\{A_i : i \in I\}$ be a collection of closed subsets of A s.t. $\bigcap_{i \in I} A_i = \phi$. Since A is closed, each A_i is closed in X. Since X is compact, $A_{i_1} \cap \dots \cap A_{i_m} = \phi$ for some $i_1, \dots, i_m \in I$. Thus A is compact. $\boxtimes$

<u>Theorem:</u> If X is Hausdorff and $A \subseteq X$, A compact then A is closed (i.e. a compact subset of a Hausdorff space is closed).

<u>Proof</u>: Take any $x \notin A$. For any $a \in A$ there are open sets U_a, V_a s.t. $a \in U_a$, $x \in V_a$ and $U_a \cap V_a = \phi$. The collection $\{U_a \cap A : a \in A\}$ is an open cover of A. Thus $A \subseteq U_{a_1} \cup \dots \cup A_{a_m}$ for some $a_1, \dots, a_m \in A$. Take $V_x = V_{a_1} \cap \dots \cap V_{a_m}$. Then V_x is open, $x \in V_x$ and $V_x \cap U_{a_i} = \phi$ for $i = 1, \dots m$. It follows that $V_x \cap (U_{a_1} \cup \dots \cup U_{a_m}) = \phi$, hence $V_x \cap A = \phi$ (as $A \subseteq U_{a_1} \cup \dots U_{a_m}$). Since $V_x \cap A = \phi$, $x \notin \overline{A}$. We proved that if $x \notin A$ then $x \notin \overline{A}$. This means $A = \overline{A}$, i.e. A is closed. $\boxtimes$