Last time we proved the following result:

<u>Thm</u>: If X is compact then any infinite subset of X has a limit point.

It turns out that for <u>metric spaces</u> the converse also holds.
In fact we will see that something stronger is true.
We will consider the following two properties:

<u>Property 1</u>: Every infinite subset of X has a limit point

<u>Property 2</u>: Every sequence in X has a convergent subsequence.

<u>Proposition 1</u>: If a topological space X has property 2 then it also has property 1, i.e. property 2 $\Longrightarrow$ property 1.

<u>Proof</u>: Let A be an infinite subset of X. Then A contains a countable ifinite subset, which then can be arrange into a sequence $a_1, a_2, \dots$ (in other words, there is an injective function $\mathbb{N} \xrightarrow[f]{} X$, $a_i = f(i)$). Since by property 2 this sequence has a convergent subsequence, we can righ away assume that $a_1, a_2, a_3, \dots$ is a convergent sequence such that $a_i \in A$ and $a_i \neq a_j$ for $i \neq j$. Since every open neighborhood of $a = \lim_{n \to \infty} a_i$ contains all but a finite number of a_i, a is a limit point of $\{a_1, a_2, \dots\}$ and hence also of A. $\boxtimes$

<u>Proposition 2</u>: If (X, d) is a metric space which has property 1, then it has property 2.

<u>Pf</u>: Consider a sequence $a_1, a_2, a_3, \dots$ in X. If this sequence has a constant subsequence (i.e. some element appears infinitely many times in the sequence), then these subsequence clearly is convergent (a constant sequence is always convergent).

Suppose now that our sequence does not have any constant subsequences. Then the set $S=\{a_1,a_2,a_3,\dots\}$ is infinite. By property 1, S has a limit point a.

Claim: If U is an open set which contains a then there are infinitely many natural numbers k s.t. $a_k\neq a$ and $a_k\in U$.

In fact, $a_k=a$ is possible for only finitely many k.

If the claim was false, we would have $a_j\notin U$ for all $j\geq N$ (for some N). If $i\leq N$ and $a_i\neq a$ then there is open set U_i s.t. $a\in U_i$ and $a_i\notin U_i$ (since X is T_1). If $a_i=a$, set $U_i=U$. Then $U_1\cap U_2\cap\dots\cap U_{N-1}\cap U=V$ is an open set s.t. $a\in V$ and if $a_s\in V$ then $a_s=a$. In other words, $V\cap S\subseteq\{a\}$, which contradicts the fact that a is a limit point of S. This proves the claim.

Now we construct a convergent subsequence as follows:

Set a_{n_1} to be any element in $B(a,1)\cap S$.

Then choose $n_2>n_1$ s.t. $a_{n_2}\in B(a,\frac{1}{2})\cap S$ (it exists by the claim).

Then choose $n_3>n_2$ s.t. $a_{n_3}\in B(a,\frac{1}{3})\cap S$ (it exists by the claim)...

This way we construct $n_1<n_2<n_3<\dots$ such that $a_{n_i}\in B(a,\frac{1}{i})\cap S$. This means that $d(a,a_{n_i})<\frac{1}{i}$ for all i, i.e. $\lim_{k\to\infty} a_{n_k}=a$. In other words, we have a convergent subsequence. ☒

Exercise: Show that Proposition 2 holds for any topological space X which is T_1 and 1st countable.

Corollary: For a metric space X properties 1 and 2 are equivalent. In particular, every compact metric space has property 2.

Definition: Let (X, d) be a metric space. A subset T of X is called an ε-net (where $\varepsilon > 0$ is a number) if the collection $\{B(t, \varepsilon) : t \in T\}$ covers X. In other words, for any $x \in X$ there is $t \in T$ s.t. $d(x, t) < \varepsilon$.

Proposition 3: If (X, d) is a metric space which has property 1 then for every $\varepsilon > 0$ there is a finite ε-net in X.

Pf. ~~Pick any~~ Suppose X does not have any finite ε-nets. Pick $a_1 \in X$. $B(a_1, \varepsilon)$ does not cover X, so there is $a_2 \notin B(a_1, X)$. Similarly, $B(a_1, \varepsilon) \cup B(a_2, \varepsilon)$ does not cover X, so there is $a_3 \notin B(a_1, \varepsilon) \cup B(a_2, \varepsilon)$. In this way we construct a sequence $a_1, a_2, a_3, \dots$, such that $a_{k+1} \notin B(a_1, \varepsilon) \cup \dots \cup B(a_k, \varepsilon)$. In other words, $d(a_i, a_j) \geq \varepsilon$ for any $i \neq j$. The set $S = \{a_1, a_2, a_3, \dots\}$ is infinite. By property 1, S has a limit point a. If $a = a_k$ for some k, then $B(a, \varepsilon) \cap S = \{a_k\}$, contrary to the definition of a limit point

If $a \notin S$, then $B(a,\frac{\varepsilon}{2})\cap S$ has at most one point. In fact, if $a_k, a_m \in B(a,\frac{\varepsilon}{2})$ and $m\neq n$ then $d(a_m,a_k)\le d(a,a_m)+d(a,a_k)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$, a contradiction.
If $B(a,\frac{\varepsilon}{2})\cap S=\emptyset$, we get a contradiction with the fact that a is a limit point of S. If $B(a,\frac{\varepsilon}{2})\cap S=\{a_n\}$ then $a\neq a_n$ (as $a\notin S$) and $B(a,r)\cap S=\emptyset$, where $r=d(a,a_n)$. This again contradicts the fact that a is a limit point of S. The contradictions show that our assumption that X has no finite ε-nets must be wrong. $\boxtimes$

<u>Proposition 4:</u> A metric space (X,d) which has property 1 is separable.

<u>Pf:</u> Let T_k be a finite $\frac{1}{k}$-net in X. Then the set $T_1\cup T_2\cup T_3\cup\ldots=\bigcup_{i=1}^{\infty}T_i$ is countable and dense. In fact, if $\varepsilon>0$ and $x\in X$ then take k s.t. $\frac{1}{k}<\varepsilon$ and note that there is $y\in T_k$ s.t. $d(x,y)<\frac{1}{k}<\varepsilon$. $\boxtimes$

<u>Proposition 5:</u> A metric space (X,d) which has property 1 is 2nd countable.

<u>Pf:</u> We proved long time ago (Thm 3.12 in the book) that a separable metric space is 2nd countable. $\boxtimes$

Proposition 6: A 2nd countable topological space X which is T_1 and has property ① is compact.

Pf: Let $\mathcal{B}$ be a countable basis of X. To prove compactness of X, it suffices to show that any covering of X by open sets from $\mathcal{B}$ has a finite subcovering. Consider any such covering. It only has countably many different open sets, so we can list them: $U_1, U_2, U_3, \ldots$. Thus $X = U_1 \cup U_2 \cup \ldots$ and we need to show that there is a finite subcover. Suppose not. Then for any $n \geq 1$, the set $U_1 \cup U_2 \cup \ldots \cup U_n$ is not X, so there is $a_n \in X$ s.t. $a_n \notin U_1 \cup U_2 \cup \ldots \cup U_n$. We have a sequence $a_1, a_2, \ldots$ with the property that for $k \geq m$ we have $a_k \notin U_m$. In particular, the set $S = \{a_1, a_2, \ldots\}$ is infinite (otherwise we would have $S \subseteq U_1 \cup \ldots \cup U_N$ for some N). By property 1, S has a limit point $a \in X$. Thus $a \in U_k$ for some k. Now $U_k \cap S \subseteq \{a_1, a_2, \ldots, a_{k-1}\}$ as $a_m \notin U_k$ for $m \geq k$. If $a_i \neq a$ there is V_i open s.t. $a \in V_i$, $a_i \notin V_i$ (property T_1). If $a_i = a$ set $V_i = U_k$. Then $V_1 \cap V_2 \cap \ldots \cap V_{k-1} \cap U_k = V$ is open, $a \in V$ and $V \cap S \subseteq \{a\}$, a contradiction with the fact that a is a limit point of S. Our assumption that $\{U_i : i = 1, 2, \ldots\}$ has no finite subcover leads to a contradiction. Thus there must exist a finite subcover. $\boxtimes$

Corollary: A metric space which satisfies (1) is compact.

Let us summarize our results:

Theorem: A metrix space is compact if and only if it has property 2: any sequence in X has a convergent subsequance. Moreover, any compact metric space is separable and 2nd countable.

Exercise: For topological spaces which are T_1 and 1st countable property 2 is equivalent to the following property: any countable covering of the space has a finite subcovering.

Remark: Problem 4 to chapter 6 in the book asks to prove that any compact space has property (2). This is FALSE. Metric compact spaces, or compact spaces which are T_1 and 1st countable have this property, but there exist counterexamples.

Theorem: Let (X,d) be a metric space. For any open cover $\{U_i : i \in I\}$ of X there is $\varepsilon > 0$ such that for every $x \in X$ there is $i \in I$ s.t. $B(x,\varepsilon) \subseteq U_i$.

Definition: Any such ε is called ~~a~~ Lebesgue number of the covering.

Proof: For any $x\in X$ there is $i(x)\in I$ such that $x\in U_{i(x)}$. Since $U_{i(x)}$ is open, there is $\varepsilon_x>0$ such that $B(x,\varepsilon_x)\subseteq U_{i(x)}$. The set $\{B(x,\frac{\varepsilon_x}{2}) : x\in X\}$ is an open cover of X. Since X is compact, we have some finite number $x_1,\dots,x_k$ of points s.t. $B(x_1,\frac{\varepsilon_{x_1}}{2}),\dots,B(x_k,\frac{\varepsilon_{x_k}}{2})$ cover X. Take $\varepsilon=\min(\frac{\varepsilon_1}{2},\dots,\frac{\varepsilon_k}{2})$. Note that if $x\in X$ then $x\in B(x_j,\frac{\varepsilon_{x_j}}{2})$ for some j. But then $B(x,\varepsilon)\subseteq B(x_j,\varepsilon_{x_j})\subseteq U_{i(x_j)}$. In fact if $d(x,a)<\varepsilon$ then $d(x_j,a)\leq d(x_j,x)+d(x,a)<\frac{\varepsilon_{x_j}}{2}+\varepsilon\leq\varepsilon_{x_j}$. $\boxtimes$

Remark: This result is on page 89 in the book and is called Lebesgue's Lemma. The proof given in the book is different, but it is worth contemplating.

Exercise: ① If (X,d) is a metric space then define $d_1(x,y)=\frac{d(x,y)}{1+d(x,y)}$. Show that d_1 is a metric on X inducing the same topology on X as d. Note: $d_1(x,y)\leq 1$ for all x,y.

② If $(X_1,d_1),(X_2,d_2),\dots$ is a sequence of metric spaces such that d_i is bounded by 1 then consider $X_1\times X_2\times X_3\times\dots$ and set $d((x_1,x_2,\dots),(y_1,y_2,\dots))=\sum_{i=1}^{\infty}\frac{d_i(x_i,y_i)}{2^i}$. Prove that d is a metric on $X_1\times X_2\times\dots$ which induces the product topology.

(3) Prove that if $X_1, X_2, \ldots$ are compact metric spaces then $X_1 \times X_2 \times \ldots$ is compact.

Of course (3) is a special case of Tychonoff Theorem

<u>Hint</u>: Consider a sequence $x_1, x_2, \ldots$ in $X_1 \times X_2 \times \ldots$ Thus $x_i = (x_{i,1}, x_{i,2}, x_{i,3}, \ldots)$. There is a subsequence such that the first coordinates converge (X_1 is compact). This subsequence has a subsequence where second coordinates converge, etc. Use this to show a subsequence which converges in $X_1 \times X_2 \times \ldots$.