MATH 461: TOPOLOGY FALL 2022

Quiz 02, August 29

Name (Print): ____________________

1. The Cantor Diagonal Argument is a method used to show that a set is uncountable. Let X be the set of sequences taken from the integers mod 5, $\mathbb{Z}_5$, so that an element of X is a sequence $\mathbf{a} = (a_n)_{n=1}^{\infty}$ where $a_n = 0, 1, 2, 3,$ or 4 for each n.

To use the Cantor Diagonal Argument to show that X is uncountable, we suppose, seeking a contradiction, that X is countable. If this were so, then there would be a complete list $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots$ of elements of X, where $\mathbf{a}_k = (a_{k,n})_{n=1}^{\infty}$.

The Diagonal Argument constructs a sequence $\mathbf{b}$ from this list such that $\mathbf{b} \notin X$.

(a) What is a formula for this sequence $\mathbf{b} = (b_n)_{n=1}^{\infty}$? (Hint: there are many possible correct answers.)

(b) How does your formula for $\mathbf{b}$ allow you to show that $\mathbf{b} \notin X$?

2. How many subsets does $\mathbb{Z}_5$ have?

3. Let A be any set and consider a function $f\colon A \to P(A)$. Let $B = \{a \in A \mid a \notin f(a)\}$. Show that there is no element $b \in A$ such that $f(b) = B$.

4. Consider the following statement about sets A, B, C, D:

$$(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$$

(a) Is this statement true for all sets A, B, C, D?

(b) If the statement is true, give a proof. If it is not true, give a counterexample.

5. Consider the following statement about sets A, B, C:

If $f \colon A \to B$ is injective and $g \colon B \to C$ is surjective, then $g \circ f \colon A \to C$ is surjective.

(a) Is this statement true for all sets A, B, C and functions f, g satisfying the conditions above?

(b) If the statement is true, give a proof. If it is not true, give a counterexample.