CHAPTER 6

More on the infinite: Products and partial fractions

Reason's last step is the recognition that there are an infinite number of things which are beyond it.
Blaise Pascal (1623–1662), Pensees. 1670.

We already met François Viète's infinite product expression for π in Section 4.10. This chapter is devoted entirely to the theory and application of infinite products, and as a consolation prize we also talk about partial fractions. In Sections 6.1 and 6.2 we present the basics of infinite products, and in Section 6.3 we look at a cool (but little publicized) theorem called *Tannery's theorem*, which is a very handy result we'll used in subsequent sections. Hold on to your seats, because the rest of the chapter is full of surprises! Are you ready to be shocked?

We begin with the following "Viète-type" formula for $\log 2$, which is due to Philipp Ludwig von Seidel (1821–1896):

$$\boxed{\log 2 = \frac{2}{1+\sqrt{2}} \cdot \frac{2}{1+\sqrt{\sqrt{2}}} \cdot \frac{2}{1+\sqrt{\sqrt{\sqrt{2}}}} \cdot \frac{2}{1+\sqrt{\sqrt{\sqrt{\sqrt{2}}}}} \cdots .}$$

Recall that if $p(z)$ is a polynomial with roots $r_1, \ldots, r_n$, then we can factor $p(z)$ as $p(z) = a(z - r_1)(z - r_2)\cdots(z - r_n)$. Euler noticed that the function $\sin \pi z$ has roots at $0, \pm 1, \pm 2, \pm 3, \ldots$, so thinking of $\sin \pi z$ as a polynomial, we have (without caring about being rigorous for the moment!)

$$\begin{aligned}\sin \pi z &= az(z-1)(z+1)(z-2)(z+2)(z-3)(z+3)\cdots \\ &= bz\Big(1 - \frac{z^2}{1^2}\Big)\Big(1 - \frac{z^2}{2^2}\Big)\Big(1 - \frac{z^2}{3^2}\Big)\cdots ,\end{aligned}$$

where a, b are constants and where we combined $(z-3)(z+3)$ into a multiple of $(1 - z^2/3^2)$ with similar remarks for the other products. In Section 6.4, we prove that Euler's guess was correct (with $b = \pi$)! Here's Euler's famous formula:

$$\boxed{\sin \pi z = \pi z\Big(1 - \frac{z^2}{1^2}\Big)\Big(1 - \frac{z^2}{2^2}\Big)\Big(1 - \frac{z^2}{3^2}\Big)\Big(1 - \frac{z^2}{4^2}\Big)\Big(1 - \frac{z^2}{5^2}\Big)\cdots .} \tag{6.1}$$

There are many applications of this result, one of which is John Wallis' infinite product expansion for π:

$$\boxed{\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots .}$$

In Section 6.5, we look at partial fraction expansions of the trig functions. Again recall that if $p(z)$ is a polynomial with roots $r_1, \ldots, r_n$, then we can factor $p(z)$ as

$p(z) = a(z - r_1)(z - r_2)\cdots(z - r_n)$, and from elementary calculus, we can write

$$\frac{1}{p(z)} = \frac{1}{a(z - r_1)(z - r_2)\cdots(z - r_n)} = \frac{a_1}{z - r_1} + \frac{a_2}{z - r_2} + \cdots + \frac{a_n}{z - r_n}.$$

You probably studied this in the "partial fraction method of integration" section in your elementary calculus course. Writing $\sin \pi z = az(z-1)(z+1)(z-2)(z+2)(z-3)(z+3)\cdots$, Euler thought that we should be able to apply this partial fraction decomposition to $1/\sin \pi z$:

$$\frac{1}{\sin \pi z} = \frac{a_1}{z} + \frac{a_2}{z-1} + \frac{a_3}{z+1} + \frac{a_4}{z-2} + \frac{a_5}{z+2} + \cdots .$$

In Section 6.5, we'll prove that this can be done where $a_1 = 1$ and $a_2 = a_3 = \cdots = -1$. That is, we'll prove that

$$\frac{\pi}{\sin \pi z} = \frac{1}{z} - \frac{1}{z-1} - \frac{1}{z+1} - \frac{1}{z-2} - \frac{1}{z+2} - \frac{1}{z-3} - \frac{1}{z+3} - \cdots .$$

Combining the adjacent factors, $-\frac{1}{z-n} - \frac{1}{z+n} = \frac{2z}{n^2-z^2}$, we get Euler's celebrated partial fraction expansion for sine:

$$\boxed{\frac{\pi}{\sin \pi z} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{n^2 - z^2}.} \tag{6.2}$$

We'll also derive partial fraction expansions for the other trig functions. In Section 6.6, we give more proofs of Euler's sum for $\pi^2/6$ using the infinite products and partial fractions we found in Sections 6.4 and 6.5. In Section 6.7, we prove one of the most famous formulas for the Riemann zeta function, namely writing it as an infinite product involving only the *prime* numbers:

$$\boxed{\zeta(z) = \frac{2^z}{2^z - 1} \cdot \frac{3^z}{3^z - 1} \cdot \frac{5^z}{5^z - 1} \cdot \frac{7^z}{7^z - 1} \cdot \frac{11^z}{11^z - 1} \cdots ,}$$

In particular, setting $z = 2$, we get the following expression for $\pi^2/6$:

$$\boxed{\frac{\pi^2}{6} = \prod \frac{p^2}{p^2 - 1} = \frac{2^2}{2^2 - 1} \cdot \frac{3^2}{3^2 - 1} \cdot \frac{5^2}{5^2 - 1} \cdots .}$$

As a bonus prize, we see how π is related to questions from probability. Finally, in Section 6.8, we derive some awe-inspiring beautiful formulas (too many to list at this moment!). Here are a couple of my favorite formulas of all time:

$$\boxed{\frac{\pi}{4} = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \frac{23}{24} \cdots .}$$

The numerators of the fractions on the right are the odd prime numbers and the denominators are even numbers divisible by four and differing from the numerators by one. The next one is also a beaut:

$$\boxed{\frac{\pi}{2} = \frac{3}{2} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \frac{23}{22} \cdots .}$$

The numerators of the fractions are the odd prime numbers and the denominators are even numbers not divisible by four and differing from the numerators by one.

Chapter 6 objectives: The student will be able to ...

- determine the (absolute) convergence for an infinite product.

- explain the infinite products and partial fraction expansions of the trig functions.
- describe Euler's formulæ for powers of π and their relationship to Riemann's zeta function.

6.1. Introduction to infinite products

We start our journey through infinite products taking careful steps to define what these phenomenal products are.

6.1.1. Basic definitions and examples. Let $\{b_n\}$ be a sequence of complex numbers. An infinite product

$$\prod_{n=1}^{\infty} b_n = b_1 \cdot b_2 \cdot b_3 \cdots$$

is said to **converge** if there exists an $m \in \mathbb{N}$ such that the b_n's are nonzero for all $n \geq m$, and the limit of partial products $\prod_{k=m}^{n} b_k = b_m \cdot b_{m+1} \cdots b_n$:

$$\lim_{n\to\infty} \prod_{k=m}^{n} b_k = \lim_{n\to\infty} \big(b_m \cdot b_{m+1} \cdots b_n\big) \tag{6.3}$$

converges to a *nonzero* complex value, say p. In this case, we define

$$\prod_{n=1}^{\infty} b_n := b_1 \cdot b_2 \cdots b_{m-1} \cdot p.$$

This definition is of course independent of the m chosen such that the b_n's are nonzero for all $n \geq m$. The infinite product $\prod_{n=1}^{\infty} b_n$ **diverges** if it doesn't converge; that is, either there are infinitely many zero b_n's or the limit (6.3) diverges or the limit (6.3) converges to zero. In this latter case, we say that the infinite product **diverges to zero**.

Example 6.1. The "harmonic product" $\prod_{n=2}^{\infty}(1-1/n)$ diverges to zero because

$$\prod_{k=2}^{n}\Big(1-\frac{1}{k}\Big) = \Big(1-\frac{1}{2}\Big)\cdots\Big(1-\frac{1}{n}\Big) = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{n-1}{n} = \frac{1}{n} \to 0.$$

Example 6.2. On the other hand, the product $\prod_{n=2}^{\infty}(1-1/n^2)$ converges because

$$\begin{aligned}\prod_{k=2}^{n}\Big(1-\frac{1}{k^2}\Big) &= \prod_{k=2}^{n}\frac{k^2-1}{k^2} = \prod_{k=2}^{n}\frac{(k-1)(k+1)}{k\cdot k}\\ &= \frac{1\cdot 3}{2\cdot 2}\cdot\frac{2\cdot 4}{3\cdot 3}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{4\cdot 6}{5\cdot 5}\cdots\frac{(n-1)(n+1)}{n\cdot n} = \frac{n+1}{2n} \to \frac{1}{2} \neq 0.\end{aligned}$$

PROPOSITION 6.1. *If an infinite product converges, then its factors tend to one. Also, a convergent infinite product has the value* 0 *if and only if it has a zero factor.*

PROOF. The second statement is automatic from the definition of convergence. If none of the b_n's vanish for $n \geq m$ and $p_n = b_m \cdot b_{m+1} \cdots b_n$, then $p_n \to p$, a nonzero number, so

$$b_n = \frac{b_m \cdot b_{m+1} \cdots b_{n-1} \cdot b_n}{b_m \cdot b_{m+1} \cdots b_{n-1}} = \frac{p_n}{p_{n-1}} \to \frac{p}{p} = 1.$$

□

Because the factors of a convergent infinite product always tend to one, we henceforth write b_n as $1 + a_n$, so the infinite product takes the form

$$\prod (1 + a_n);$$

then this infinite product converges implies that $a_n \to 0$.

6.1.2. Infinite products and series: the nonnegative case. The following theorem states that the analysis of an infinite product $\prod(1 + a_n)$ with all the a_n's nonnegative is completely determined by the infinite series $\sum a_n$.

THEOREM 6.2. *An infinite product* $\prod(1 + a_n)$ *with nonnegative terms* a_n *converges if and only if the series* $\sum a_n$ *converges.*

PROOF. Let the partial products and partial sums be denoted by

$$p_n = \prod_{k=1}^{n} (1 + a_k) \quad \text{and} \quad s_n = \sum_{k=1}^{n} a_k.$$

Since all the a_k's are nonnegative, both sequences $\{p_n\}$ and $\{s_n\}$ are nondecreasing, so converge if and only if they are bounded. Since $1 + x \le e^x$ for any real number x (see Theorem 4.30), it follows that

$$p_n = \prod_{k=1}^{n} (1 + a_k) \le \prod_{k=1}^{n} e^{a_k} = e^{\sum_{k=1}^{n} a_k} = e^{s_n}.$$

This equation shows that if the sequence $\{s_n\}$ is bounded, then the sequence $\{p_n\}$ is also bounded. On the other hand,

$$p_n = (1 + a_1)(1 + a_2) \cdots (1 + a_n) \ge 1 + s_n,$$

since the left-hand side, when multiplied out, contains the sum $1 + a_1 + a_2 + \cdots + a_n$ (and a lot of other nonnegative terms too). This shows that if the sequence $\{p_n\}$ is bounded, then the sequence $\{s_n\}$ is also bounded. □

Example 6.3. Thus, as a consequence of this theorem, the product

$$\prod \left(1 + \frac{1}{n^p}\right)$$

converges for $p > 1$ and diverges for $p \le 1$.

6.1.3. An infinite product for $\log 2$ **and** e**.** I found the following gem in [**150**]. Define a sequence $\{e_n\}$ by $e_1 = 1$ and $e_{n+1} = (n+1)(e_n+1)$ for $n = 1, 2, 3, \ldots$; e.g.

$$e_1 = 1\ ,\ e_2 = 4\ ,\ e_3 = 15\ ,\ e_4 = 64\ ,\ e_5 = 325\ ,\ e_6 = 1956\ , \ldots.$$

Then

$$e = \prod_{n=1}^{\infty} \frac{e_n + 1}{e_n} = \frac{2}{1} \cdot \frac{5}{4} \cdot \frac{16}{15} \cdot \frac{65}{64} \cdot \frac{326}{325} \cdot \frac{1957}{1956} \cdots . \tag{6.4}$$

You will be asked to prove this in Problem 6.

We now prove Philipp Ludwig von Seidel's (1821–1896) formula for $\log 2$:

$$\log 2 = \frac{2}{1 + \sqrt{2}} \cdot \frac{2}{1 + \sqrt{\sqrt{2}}} \cdot \frac{2}{1 + \sqrt{\sqrt{\sqrt{2}}}} \cdot \frac{2}{1 + \sqrt{\sqrt{\sqrt{\sqrt{2}}}}} \cdots .$$

To prove this, we follow the proof of Viète's formula in Section 4.10.6 using hyperbolic functions instead of trigonometric functions. Let $x \in \mathbb{R}$ be nonzero. Then dividing the identity $\sinh x = 2\cosh(x/2)\sinh(x/2)$ (see Problem 8 in Exercises 4.7) by x, we get

$$\frac{\sinh x}{x} = \cosh(x/2) \cdot \frac{\sinh(x/2)}{x/2}.$$

Replacing x with $x/2$, we get $\sinh(x/2)/(x/2) = \cosh(x/2^2) \cdot \sinh(x/2^2)/(x/2^2)$, therefore

$$\frac{\sinh x}{x} = \cosh(x/2) \cdot \cosh(x/2^2) \cdot \frac{\sinh(x/2^2)}{x/2^2}.$$

Continuing by induction, we obtain for any $n \in \mathbb{N}$,

$$\frac{\sinh x}{x} = \prod_{k=1}^{n} \cosh(x/2^k) \cdot \frac{\sinh(x/2^n)}{x/2^n},$$

or

$$\frac{x}{\sinh x} \cdot \frac{\sinh(x/2^n)}{x/2^n} = \prod_{k=1}^{n} \frac{1}{\cosh(x/2^k)}.$$

Taking $n \to \infty$, it follows that

$$\frac{x}{\sinh x} = \lim_{n\to\infty} \frac{x}{\sinh x} \cdot \frac{\sinh(x/2^n)}{x/2^n} = \lim_{n\to\infty} \prod_{k=1}^{n} \frac{1}{\cosh(x/2^k)} \tag{6.5}$$

since $\lim_{n\to\infty} \frac{\sinh(x/2^n)}{x/2^n} = 1$ for any nonzero $x \in \mathbb{R}$ (why?). Now let us put $x = \log\theta$, that is, $\theta = e^x$, into the equation (6.5). To this end, observe that

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{\theta - \theta^{-1}}{2} = \frac{\theta^2 - 1}{2\theta} \quad \Longrightarrow \quad \frac{x}{\sinh x} = \frac{2\theta \log\theta}{(\theta-1)(\theta+1)}$$

and

$$\cosh(x/2^k) = \frac{e^{\frac{x}{2^k}} + e^{-\frac{x}{2^k}}}{2} = \frac{\theta^{\frac{1}{2^k}} + \theta^{-\frac{1}{2^k}}}{2} = \frac{\theta^{\frac{1}{2^{k-1}}} + 1}{2\theta^{\frac{1}{2^k}}}$$

$$\Longrightarrow \quad \frac{1}{\cosh(x/2^k)} = \frac{2\,\theta^{\frac{1}{2^k}}}{\theta^{1/2^{k-1}} + 1}.$$

Thus,

$$\begin{aligned}\frac{2\theta\log\theta}{(\theta-1)(\theta+1)} = \lim_{n\to\infty} \prod_{k=1}^{n} \frac{2\,\theta^{\frac{1}{2^k}}}{\theta^{1/2^{k-1}} + 1} &= \lim_{n\to\infty}\left(\prod_{k=1}^{n} \theta^{\frac{1}{2^k}} \cdot \prod_{k=1}^{n} \frac{2}{\theta^{\frac{1}{2^{k-1}}} + 1}\right)\\ &= \lim_{n\to\infty}\left(\theta^{\sum_{k=1}^{n}\frac{1}{2^k}} \cdot \prod_{k=1}^{n} \frac{2}{\theta^{\frac{1}{2^{k-1}}} + 1}\right).\end{aligned}$$

Since $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2^k} = 1$ (this is just the geometric series $\sum_{k=1}^{\infty} \frac{1}{2^k}$), we see that

$$\frac{2\theta\log\theta}{(\theta-1)(\theta+1)} = \theta \cdot \lim_{n\to\infty} \prod_{k=1}^{n} \frac{2}{\theta^{\frac{1}{2^{k-1}}} + 1} = \theta \cdot \frac{2}{\theta+1} \cdot \lim_{n\to\infty} \prod_{k=1}^{n-1} \frac{2}{\theta^{\frac{1}{2^k}} + 1}.$$

Cancelling like terms, we have, by definition of infinite products, the following beautiful infinite product expansion for $\frac{\log\theta}{\theta-1}$:

$$\frac{\log\theta}{\theta-1} = \prod_{k=1}^{\infty} \frac{2}{1+\theta^{\frac{1}{2^k}}} = \frac{2}{1+\sqrt{\theta}} \cdot \frac{2}{1+\sqrt{\sqrt{\theta}}} \cdot \frac{2}{1+\sqrt{\sqrt{\sqrt{\theta}}}} \cdots \textit{\textbf{Seidel's formula}}.$$

In particular, taking $\theta = 2$, we get Seidel's infinite product formula for $\log 2$.

Exercises 6.1.

1. Prove that

$$(a)\ \prod_{n=2}^{\infty}\Big(1-\frac{2n+1}{n(n+2)}\Big) = 3,\quad (b)\ \prod_{n=3}^{\infty}\Big(1-\frac{2}{n(n-1)}\Big) = \frac{1}{3},$$

$$(c)\ \prod_{n=2}^{\infty}\Big(1+\frac{2n+1}{n^2-1}\Big) = \frac{1}{3},\quad (d)\ \prod_{n=2}^{\infty}\Big(1+\frac{(-1)^n}{n}\Big) = 1.$$

2. Prove that for any $z \in \mathbb{C}$ with $|z| < 1$,

$$\prod_{n=0}^{\infty}\Big(1+z^{2^n}\Big) = \frac{1}{1-z}.$$

Suggestion: Derive, e.g. by induction, a formula for $p_n = \prod_{k=0}^{n}(1+z^{2^k})$ as a geometric sum. For example, we have the neat formula $\prod_{n=0}^{\infty}\Big(1+\big(\frac{1}{2}\big)^{2^n}\Big) = 2$.

3. Determine convergence for:

$$(a)\ \prod_{n=1}^{\infty}\Big(1+\sin^2\Big(\frac{1}{n}\Big)\Big)\quad,\quad (b)\ \prod_{n=1}^{\infty}\Big(1+\Big(\frac{nx^2}{1+n}\Big)^n\Big)\quad,\quad (c)\ \prod_{n=1}^{\infty}\Big(\frac{1+x^2+x^{2n}}{1+x^{2n}}\Big),$$

where for (b) and (c), state for which $x \in \mathbb{R}$, the products converge and diverge.

4. In this problem, we prove that an infinite product $\prod(1-a_n)$ with $0 \le a_n < 1$ converges if and only if the series $\sum a_n$ converges.
   (i) Let $p_n = \prod_{k=1}^{n}(1-a_k)$ and $s_n = \sum_{k=1}^{n} a_k$. Show that $p_n \le e^{-s_n}$. Conclude that if $\sum a_n$ diverges, then $\prod(1-a_n)$ also diverges (in this case, diverges to zero).
   (ii) Suppose now that $\sum a_n$ converges. Then we can choose m such that $a_m + a_{m+1} + \cdots < 1/2$. Prove by induction that

$$(1-a_m)(1-a_{m+1})\cdots(1-a_n) \ge 1-(a_m+a_{m+1}+\cdots+a_n)$$

   for $n = m, m+1, m+2, \ldots$. Conclude that $p_n/p_m \ge 1/2$ for all $n \ge m$, and from this, prove that $\prod(1-a_n)$ converges.
   (iii) For what p is $\prod_{n=2}^{\infty}\big(1-\frac{1}{n^p}\big)$ convergent and divergent?

5. In this problem we derive relationships between series and products. Let $\{a_n\}$ be a sequence of complex numbers with $a_n \neq 0$ for all n.
   (a) Prove that for $n \ge 2$,

$$\prod_{k=1}^{n}(1+a_k) = a_1 + \sum_{k=2}^{n}(1+a_1)\cdots(1+a_{k-1})a_k.$$

   Thus, $\prod_{n=1}^{\infty}(1+a_n)$ converges if and only if $a_1 + \sum_{k=2}^{\infty}(1+a_1)\cdots(1+a_{k-1})a_k$ converges to a nonzero value, in which case they have the same value.
   (b) Assume that $a_1 + \cdots + a_k \neq 0$ for every k. Prove that for $n \ge 2$,

$$\sum_{k=1}^{n} a_k = a_1 \prod_{k=2}^{n}\left(1+\frac{a_k}{a_1+a_2+\cdots+a_{k-1}}\right).$$

Thus, $\sum_{n=1}^{\infty} a_n$ converges if and only if $a_1 \prod_{n=2}^{\infty} \left(1 + \frac{a_n}{a_1+a_2+\cdots+a_{n-1}}\right)$ either converges or diverges to zero, in which case they have the same value.

(c) Using (b) and the sum $\sum_{n=1}^{\infty} \frac{1}{(n+a-1)(n+a)} = \frac{1}{a}$ from (3.36), prove that

$$\prod_{n=2}^{\infty} \left(1 + \frac{a}{(n+a)(n-1)}\right) = a + 1.$$

6. In this problem we prove (6.4)
   (i) Let $s_n = \sum_{k=0}^{n} \frac{1}{k!}$. Prove that $e_n = n!\, s_{n-1}$ for $n = 1, 2, \ldots$.
   (ii) Show that $s_n/s_{n-1} = (e_n + 1)/e_n$.
   (iii) Show that $s_n = \prod_{k=1}^{n} \frac{e_k+1}{e_k}$ and then complete the proof. Suggestion: Note that we can write $s_n = (s_1/s_0) \cdot (s_2/s_1) \cdots (s_n/s_{n-1})$.

6.2. Absolute convergence for infinite products

Way back in Section 3.6 we introduced absolute convergence for infinite series and since then we have experienced how incredibly useful this notion is. In this section we continue our study of the basic properties of infinite products by introducing a notion of absolute convergence for infinite products. We begin by presenting a general convergence test that is able to test the convergence of any infinite product in terms of a corresponding series of logarithms.

6.2.1. Infinite products and series: the general case. In Theorem 6.2, we gave a criterion for the convergence of an infinite product in terms of a corresponding series. Now what about the case for general a_n? In this general case, the infinite product is completely determined by the convergence of a corresponding infinite series of logarithms. Moreover, we also get a formula for the product in terms of the sum of the infinite series.

THEOREM 6.3. *An infinite product* $\prod(1 + a_n)$ *converges if and only if* $a_n \to 0$ *and the series*

$$\sum_{n=m+1}^{\infty} \operatorname{Log}(1 + a_n),$$

starting from a suitable index $m + 1$*, converges. Moreover, if* L *is the sum of the series, then*

$$\prod(1 + a_n) = (1 + a_1) \cdots (1 + a_m)\, e^L.$$

PROOF. First of all, we remark that the statement "starting from a suitable index $m + 1$" concerning the sum of logarithms is needed because we need to make sure the sum starts sufficiently high so that none of the terms $1 + a_n$ is zero (otherwise $\operatorname{Log}(1 + a_n)$ is undefined). By Proposition 6.1, in order for the product $\prod(1+a_n)$ to converge, we at least need $a_n \to 0$. Thus, we may assume that $a_n \to 0$; in particular we can fix m such that $n > m$ implies $|a_n| < 1$.

Let $b_n = 1 + a_n$. We shall prove that the infinite product $\prod b_n$ converges if and only if the series

$$\sum_{n=m+1}^{\infty} \operatorname{Log} b_n,$$

converges, and if L is the sum of the series, then

$$\prod b_n = b_1 \cdots b_m\, e^L. \tag{6.6}$$

For $n > m$, let the partial product and partial sums be denoted by

$$p_n = \prod_{k=m+1}^{n} b_k \qquad \text{and} \qquad s_n = \sum_{k=m+1}^{n} \operatorname{Log} b_k.$$

Since $\exp(\operatorname{Log} z) = z$ for any nonzero complex number z, it follows that

$$\exp(s_n) = p_n. \tag{6.7}$$

Thus, if the sum s_n converges to a value L, this equation shows that p_n converges to e^L, which is nonzero, and also proves the formula (6.6).

Conversely, suppose that $\{p_n\}$ converges to a nonzero complex number p. We shall prove that $\{s_n\}$ also converges; once this is established, the formula (6.6) follows from (6.7). Note that replacing b_{m+1} by b_{m+1}/p, we may assume that $p = 1$. For $k > m$, we can write $p_n = \exp(\operatorname{Log} p_n)$, so the formula (6.7) implies that for $n > m$,

$$s_n = \operatorname{Log} p_n + 2\pi i k_n$$

for some integer k_n. Moreover, since

$$s_n - s_{n-1} = \sum_{k=m+1}^{n} \operatorname{Log} b_k - \sum_{k=m+1}^{n-1} \operatorname{Log} b_k = \operatorname{Log} b_n,$$

and $b_n \to 1$ (since $a_n \to 0$), it follows that

$$\operatorname{Log} p_n - \operatorname{Log} p_{n-1} + 2\pi(k_n - k_{n-1}) = s_n - s_{n-1} \to 0$$

as $n \to \infty$. By assumption $p_n \to 1$, so we must have $k_n - k_{n-1} \to 0$, which can happen only if $k_n = k$, a fixed integer for n sufficiently large. It follows that

$$s_n = \operatorname{Log} p_n + 2\pi i k_n \to 2\pi i k,$$

which shows that $\{s_n\}$ converges. □

6.2.2. Absolute convergence. In view of Theorem 6.3, the following definition of absolute convergence for infinite products seems very natural: An infinite product $\prod(1 + a_n)$ is said to **converge absolutely** if the series

$$\sum_{n=m+1}^{\infty} \operatorname{Log}(1 + a_n),$$

starting from a suitable index $m + 1$, is absolutely convergent. It turns out that $\prod(1 + a_n)$ is absolutely convergent if and only if $\sum a_n$ is absolutely convergent, a fact we prove in Theorem 6.5 below. To this end, we first prove the following.

LEMMA 6.4. *For any complex number z with $|z| \leq 1/2$, we have*

$$\frac{1}{2}|z| \leq |\operatorname{Log}(1 + z)| \leq \frac{3}{2}|z|.$$

PROOF. Since

$$\operatorname{Log}(1 + z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n + 1},$$

for $|z| \leq 1/2$, we have

$$\left| \frac{\operatorname{Log}(1 + z)}{z} - 1 \right| \leq \sum_{n=1}^{\infty} \frac{|z|^n}{n + 1} \leq \sum_{n=1}^{\infty} \frac{1}{2^n(n + 1)} \leq \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2}.$$

This shows that
$$1-\frac{1}{2}=\frac{1}{2}\le\left|\frac{\mathrm{Log}(1+z)}{z}\right|\le 1+\frac{1}{2}=\frac{3}{2}.$$
Multiplying by $|z|$ completes the proof. □

THEOREM 6.5. *An infinite product* $\prod(1+a_n)$ *converges absolutely if and only if the sum* $\sum a_n$ *converges absolutely.*

PROOF. Since the product and sum both diverge if a_n does not tend to 0, we may assume that $a_n \to 0$, in which case we may assume that $|a_n| \le 1/2$ for all n. In particular, by our lemma, we have
$$\frac{1}{2}|a_n| \le |\,\mathrm{Log}(1+a_n)| \le \frac{3}{2}|a_n|$$
for every n. Our series comparison test now immediately implies that the sum $\sum |\,\mathrm{Log}(1+a_n)|$ converges if and only if the sum $\sum |a_n|$ converges. □

EXERCISES 6.2.

1. For what $z \in \mathbb{C}$ are the following products absolutely convergent?
$$(a)\ \prod_{n=1}^{\infty}\Big(1+z^n\Big) \quad , \quad (b)\ \prod_{n=1}^{\infty}\Big(1+\Big(\frac{nz}{1+n}\Big)^n\Big)$$
$$(c)\ \prod_{n=1}^{\infty}\Big(1+\sin^2\Big(\frac{z}{n}\Big)\Big) \quad , \quad (d)\ \prod_{n=1}^{\infty}\Big(1+\frac{z^n}{n\log n}\Big).$$
2. Here is a nice convergence test: Suppose that $\sum a_n^2$ converges. Then $\prod(1+a_n)$ converges if and only if the series $\sum a_n$ converges. You may proceed as follows.
   (i) Since $\sum a_n^2$ converges, we know that $a_n \to 0$, so we may henceforth assume that $|a_n|^2 < \frac{1}{2}$ for all n. Prove that
   $$\big|\,\mathrm{Log}(1+a_n)-a_n\big| \le |a_n|^2.$$
   Suggestion: You will need the power series expansion for $\mathrm{Log}(1+z)$.
   (ii) Prove that $\sum(\mathrm{Log}(1+a_n)-a_n)$ is absolutely convergent.
   (iii) Prove that $\sum a_n$ converges if and only if $\sum \mathrm{Log}(1+a_n)$ converges and from this, prove the desired result.
   (iv) Does the product $\prod_{n=2}^{\infty}\big(1+\frac{(-1)^n}{n}\big)$ converge? What about the product
   $$\Big(1+\frac{1}{2}\Big)\Big(1+\frac{1}{3}\Big)\Big(1-\frac{1}{4}\Big)\Big(1+\frac{1}{5}\Big)\Big(1+\frac{1}{6}\Big)\Big(1-\frac{1}{7}\Big)\Big(1+\frac{1}{8}\Big)\Big(1+\frac{1}{9}\Big)\cdots?$$
3. Let $\{a_n\}$ be a sequence of real numbers and assume that $\sum a_n$ converges but $\sum a_n^2$ diverges. In this problem we shall prove that $\prod(1+a_n)$ diverges.
   (i) Prove that there is a constant $C > 0$ such that for all $x \in \mathbb{R}$ with $|x| < 1$, we have
   $$x-\log(1+x) \ge Cx^2.$$
   (ii) Since $\sum a_n$ converges, we know that $a_n \to 0$, so we may assume that $|a_n|^2 < 1$ for all n. Using (i), prove that $\sum \log(1+a_n)$ diverges and hence, $\prod(1+a_n)$ diverges.
   (iii) Does $\prod(1+\frac{(-1)^{n-1}}{\sqrt{n}})$ converge or diverge?
4. Given a sequence of complex numbers $\{a_n\}$, prove that $\prod(1+a_n)$ converges absolutely if and only if $\prod(1+|a_n|)$ converges.
5. Using the formulas from Problem 5 in Exercises 5.9, prove that for $|z| < 1$,
$$\prod_{n=1}^{\infty}(1-z^n)=\exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\frac{z^n}{1-z^n}\right) \quad , \quad \prod_{n=1}^{\infty}(1+z^n)=\exp\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\frac{z^n}{1-z^n}\right).$$

6.3. Tannery's theorem and the exponential function (again!)

Tannery's theorem (named after Jules Tannery (1848–1910)) is a little known, but fantastic theorem, that I learned from [**24**], [**23**], [**31**, p. 136]. Tannery's theorem is really a special case of the Weierstrass M-test that we'll study in Section 8.8. We shall use Tannery's theorem quite a bit in the next two sections to derive Euler's trigonometric infinite product expansions and partial fraction expansions.

6.3.1. Tannery's theorem for series.

Tannery has two theorems, one for series and the other for products. Here is the one for series.

THEOREM 6.6 (**Tannery's theorem for series**). *For each natural number n, let $\sum_{k=1}^\infty a_k(n)$ be a convergent series. If for each k, $\lim_{n\to\infty} a_k(n) = a_k$ and $|a_k(n)| \le M_k$ for all n where the series $\sum_{k=1}^\infty M_k$ converges, then*

$$\lim_{n\to\infty} \sum_{k=1}^\infty a_k(n) = \sum_{k=1}^\infty a_k.$$

PROOF. Let $\varepsilon > 0$. Then by Cauchy's criterion for series there is an m_1 so that

$$M_{m_1+1} + M_{m_1+2} + \cdots < \frac{\varepsilon}{3}.$$

Since for any n, k, $|a_k(n)| \le M_k$, taking $n \to \infty$, we also have, for every k, $|a_k| \le M_k$ as well. Thus, using that $|a_k(n) - a_k| \le |a_k(n)| + |a_k| \le M_k + M_k = 2M_k$, we obtain

$$\begin{aligned}\left|\sum_{k=1}^\infty a_k(n) - \sum_{k=1}^\infty a_k\right| &= \left|\sum_{k=1}^{m_1}(a_k(n) - a_k) + \sum_{k=m_1+1}^\infty (a_k(n) - a_k)\right| \\ &\le \sum_{k=1}^{m_1} |a_k(n) - a_k| + \sum_{k=m_1+1}^\infty 2M_k < \sum_{k=1}^{m_1} |a_k(n) - a_k| + 2\frac{\varepsilon}{3}.\end{aligned}$$

Since for each k, $\lim_{n\to\infty} a_k(n) = a_k$, there is an N such that for each $k = 1, 2, \ldots, m_1$ and for $n > N$, we have $|a_k(n) - a_k| < \varepsilon/(3m_1)$. Thus, if $n > N$, then

$$\left|\sum_{k=1}^\infty a_k(n) - \sum_{k=1}^\infty a_k\right| < \sum_{k=1}^{m_1} \frac{\varepsilon}{3m_1} + 2\frac{\varepsilon}{3} = \varepsilon.$$

This completes the proof. □

Notice that we can write the conclusion of Tannery's theorem as

$$\lim_{n\to\infty} \sum_{k=1}^\infty a_k(n) = \sum_{k=1}^\infty \lim_{n\to\infty} a_k(n),$$

so Tannery's theorem states that under certain conditions we can switch limits and *infinite* summations. (Of course, we can always switch limits and *finite* summations by the algebra of limits, but infinite limits is a whole other matter.)

Example 6.4. As a neat application of Tannery's theorem, we shall prove the pretty formula

$$\boxed{\frac{e}{e-1} = \lim_{n\to\infty} \left\{ \left(\frac{n}{n}\right)^n + \left(\frac{n-1}{n}\right)^n + \cdots + \left(\frac{1}{n}\right)^n \right\}.}$$

To prove this, we write the right-hand side as

$$\lim_{n\to\infty}\left\{\left(\frac{n}{n}\right)^n+\left(\frac{n-1}{n}\right)^n+\cdots+\left(\frac{1}{n}\right)^n\right\}=\lim_{n\to\infty}\sum_{k=0}^{\infty}a_k(n),$$

where $a_k(n):=0$ for $k\geq n$ and for $0\leq k\leq n-1$,

$$a_k(n):=\left(\frac{n-k}{n}\right)^n=\left(1-\frac{k}{n}\right)^n.$$

Observe that

$$\lim_{n\to\infty}a_k(n)=\lim_{n\to\infty}\left(1-\frac{k}{n}\right)^n=e^{-k}=:a_k.$$

Also, for $k\leq n-1$,

$$|a_k(n)|=\left(1-\frac{k}{n}\right)^n\leq\left(e^{-k/n}\right)^n=e^{-k},$$

where we used that $1+x\leq e^x$ for all $x\in\mathbb{R}$ from Theorem 4.30. Since $a_k(n)=0$ for $k\geq n$, it follows that $|a_k(n)|\leq M_k$ for all n where $M_k=e^{-k}$. Since $\sum M_k<\infty$, by Tannery's theorem, we have

$$\lim_{n\to\infty}\left\{\left(\frac{n}{n}\right)^n+\left(\frac{n-1}{n}\right)^n+\cdots+\left(\frac{1}{n}\right)^n\right\}$$
$$=\lim_{n\to\infty}\sum_{k=0}^{\infty}a_k(n)=\sum_{k=0}^{\infty}a_k=\sum_{k=0}^{\infty}e^{-k}=\frac{1}{1-1/e}=\frac{e}{e-1}.$$

6.3.2. The exponential function (again!) Tannery's theorem gives a quick proof of Theorem 3.30, that for any sequence $z_n\to z$, we have

$$\exp(z)=\lim_{n\to\infty}\left(1+\frac{z_n}{n}\right)^n.$$

Indeed, upon expanding using the binomial theorem, we see that

$$\left(1+\frac{z_n}{n}\right)^n=1+z_n+\frac{1}{2!}\left(1-\frac{1}{n}\right)z_n^2+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)z_n^3+\cdots$$
$$+\cdots+\frac{1}{n!}\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)\right]z_n^n=\sum_{k=0}^{\infty}a_k(n),$$

where $a_k(n):=0$ for $k\geq n+1$, $a_0(n):=1$, and for $1\leq k\leq n$,

$$a_k(n):=\frac{1}{k!}\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\right]z_n^k.$$

Notice that

$$\lim_{n\to\infty}a_k(n)=\lim_{n\to\infty}\frac{1}{k!}\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\right]z_n^k=\frac{z^k}{k!}=:a_k.$$

Also notice that $|a_0(n)|=1=:M_0$ and if C is a bound on the convergent sequence $\{z_n\}$, then for $0\leq k\leq n$,

$$|a_k(n)|=\left|\frac{1}{k!}\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\right]z_n^k\right|$$
$$\leq\frac{1}{k!}\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\right]C^k\leq\frac{C^k}{k!},$$

where we used that the term in brackets is product of positive numbers ≤ 1 so the product is also ≤ 1. Since $a_k(n) = 0$ for $k \ge n+1$, it follows that $|a_k(n)| \le M_k$ for all n where $M_k = C^k/k!$. Since $\sum M_k < \infty$, by Tannery's theorem, we have

$$\lim_{n\to\infty}\left(1+\frac{z_n}{n}\right)^n = \lim_{n\to\infty}\sum_{k=0}^{\infty} a_k(n) = \sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty}\frac{z^k}{k!} = \exp(z).$$

Tannery's theorem can also be used to establish formulas for sine and cosine, see Problem 2.

6.3.3. Tannery's theorem for products. We now give a corresponding theorem for infinite products.

THEOREM 6.7 (**Tannery's theorem for infinite products**). *For each natural number n, let $\prod_{k=1}^{\infty}(1+a_k(n))$ be a convergent infinite product. If for each k, $\lim_{n\to\infty} a_k(n) = a_k$ and $|a_k(n)| \le M_k$ for all n, where the series $\sum M_k$ converges, then*

$$\lim_{n\to\infty}\prod_{k=1}^{\infty}(1+a_k(n)) = \prod_{k=1}^{\infty}(1+a_k).$$

PROOF. First of all, we choose m such that for all $k > m$, we have $M_k < 1/2$. For $n > m$, we can write

$$p(n) = q(n)\cdot\prod_{k=m+1}^{\infty}(1+a_k(n)),$$

where

$$q(n) = \prod_{k=1}^{m}(1+a_k(n)).$$

Then $q(n) \to \prod_{k=1}^{m}(1+a_k)$ as $n\to\infty$, so we're left to show that $\prod_{k=m+1}^{\infty}(1+a_k(n))$ converges to $\prod_{k=m+1}^{\infty}(1+a_k)$ as $n\to\infty$. Since $M_k < 1/2$ for $k > m$, we have $|a_k(n)| \le M_k < 1/2$ for $k > m$, so according to Lemma 6.4, we see that for $k > m$,

$$|\operatorname{Log}(1+a_k(n))| \le \frac{3}{2}|a_k(n)| \le \frac{3}{2}M_k.$$

Since $\sum_{k=1}^{\infty} M_k < \infty$, Tannery's theorem for series implies that

$$\lim_{n\to\infty}\sum_{k=m+1}^{\infty}\operatorname{Log}(1+a_k(n)) = \sum_{k=m+1}^{\infty}\operatorname{Log}(1+a_k).$$

On the other hand, by Theorem 6.3, we have

$$\prod_{k=m+1}^{\infty}(1+a_k(n)) = \exp\left(\sum_{k=m+1}^{\infty}\operatorname{Log}(1+a_k(n))\right).$$

Taking $n\to\infty$ of both sides of this equality and using Theorem 6.3 again, we obtain

$$\begin{aligned}\lim_{n\to\infty}\prod_{k=m+1}^{\infty}(1+a_k(n)) &= \lim_{n\to\infty}\exp\left(\sum_{k=m+1}^{\infty}\operatorname{Log}(1+a_k(n))\right)\\ &= \exp\left(\sum_{k=m+1}^{\infty}\operatorname{Log}(1+a_k)\right) = \prod_{k=m+1}^{\infty}(1+a_k).\end{aligned}$$

□

EXERCISES 6.3.

1. Determine the following limits.

$$(a)\ \lim_{n\to\infty}\left\{\frac{n^2}{1+(1\cdot n)^2}+\frac{n^2}{1+(2\cdot n)^2}+\cdots+\frac{n^2}{1+(n\cdot n)^2}\right\},$$

$$(b)\ \lim_{n\to\infty}\left\{\left(1-\frac{1}{4n^2\log\left(1+\left(\frac{2}{2n}\right)^2\right)}\right)\cdot\left(1-\frac{1}{4n^2\log\left(1+\left(\frac{3}{2n}\right)^2\right)}\right)\cdot\left(1-\frac{1}{4n^2\log\left(1+\left(\frac{4}{2n}\right)^2\right)}\right)\cdots\left(1-\frac{1}{4n^2\log\left(1+\left(\frac{n}{2n}\right)^2\right)}\right)\right\},$$

$$(c)\ \lim_{n\to\infty}\left\{\frac{1}{n^3\sin\left(\frac{1\cdot 2}{n^3}\right)}+\frac{1}{n^3\sin\left(\frac{2\cdot 3}{n^3}\right)}+\cdots+\frac{1}{n^3\sin\left(\frac{n\cdot(n+1)}{n^3}\right)}\right\}.$$

2. Let us suppose that we had *defined*

$$\cos z := \lim_{n\to\infty}\frac{1}{2}\left\{\left(1+\frac{iz}{n}\right)^n+\left(1-\frac{iz}{n}\right)^n\right\},\qquad z\in\mathbb{C}.$$

(This is motivated by the identity $\cos z = \frac{1}{2}\left\{e^{iz}+e^{-iz}\right\}$.) Use Tannery's theorem to prove that

$$\cos z = \sum_{k=0}^{\infty}(-1)^k\frac{z^{2k}}{(2k)!},$$

the same formula we of course already know! Similarly, let us suppose we had *defined*

$$\sin z := \lim_{n\to\infty}\frac{1}{2i}\left\{\left(1+\frac{iz}{n}\right)^n-\left(1-\frac{iz}{n}\right)^n\right\},\qquad z\in\mathbb{C}.$$

Use Tannery's theorem to prove that $\sin z = \sum_{k=0}^{\infty}(-1)^k\frac{z^{2k+1}}{(2k+1)!}$.

6.4. Euler's trigonometric expansions and π as an infinite product

The goal of this section is to prove Euler's celebrated formula (6.1) stated in the introduction of this chapter:

THEOREM 6.8 (**Euler's product for sine**). *For any complex z, we have*

$$\boxed{\sin\pi z = \pi z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right).}$$

We give two proofs of this astounding result. We also prove Wallis' infinite product expansion for π.

6.4.1. Expansion of sine I. (Cf. [**31**, p. 294]). Our first proof of Euler's infinite product for sine is based on a neat identity involving tangents that we'll present in Lemma 6.9 below. We beginning our proof by writing

$$\sin z = \frac{1}{2i}\left(e^{iz}-e^{-iz}\right)=\lim_{n\to\infty}\frac{1}{2i}\left\{\left(1+\frac{iz}{n}\right)^n-\left(1-\frac{iz}{n}\right)^n\right\}=\lim_{n\to\infty}F_n(z),$$

where F_n is the polynomial of degree n in z given by

$$F_n(z)=\frac{1}{2i}\left\{\left(1+\frac{iz}{n}\right)^n-\left(1-\frac{iz}{n}\right)^n\right\}. \tag{6.8}$$

In the following lemma, we write $F_n(z)$ in terms of tangents.

LEMMA 6.9. *If* $n = 2m+1$ *with* $m \in \mathbb{N}$, *then we can write*

$$F_n(z) = z \prod_{k=1}^{m} \left(1 - \frac{z^2}{n^2 \tan^2(k\pi/n)}\right).$$

PROOF. Observe that setting $z = n\tan\theta$, we have

$$1 + \frac{iz}{n} = 1 + i\tan\theta = 1 + i\frac{\sin\theta}{\cos\theta} = \sec\theta\, e^{i\theta},$$

and similarly, $1 - iz/n = \sec\theta\, e^{-i\theta}$. Thus,

$$F_n(n\tan\theta) = \frac{1}{2i}\sec^n\theta\Big(e^{in\theta} - e^{-in\theta}\Big) = \sec^n\theta\,\sin(n\theta).$$

It follows that $F_n(n\tan\theta) = 0$ for $n\theta = k\pi$ for any integer k, that is, for any integer k, we have $F_n(z_k) = 0$ for $z_k = n\tan(k\pi/n) = n\tan(k\pi/(2m+1))$, where we recall that $n = 2m+1$. Since $\tan\theta$ is strictly increasing and odd on $(-\pi/2, \pi/2)$, it follows that the $n = 2m+1$ values of z_k for $k = 0, \pm 1, \pm 2, \ldots, \pm m$ are distinct. Hence, we have found n distinct roots of $F_n(z)$, so as a consequence of the fundamental theorem of algebra, we can write $F_n(z)$ as a constant times

$$(z - z_0)\cdot(z - z_1)\cdots(z - z_m)\cdot(z - z_{-1})\cdots(z - z_{-m}) =$$
$$z\prod_{k=1}^{m}\left(z - n\tan\left(\frac{k\pi}{n}\right)\right)\prod_{k=1}^{m}\left(z + n\tan\left(\frac{k\pi}{n}\right)\right) = z\prod_{k=1}^{m}\left(z^2 - n^2\tan^2\left(\frac{k\pi}{n}\right)\right).$$

We can rewrite this product as

$$F_n(z) = az\prod_{k=1}^{m}\left(1 - \frac{z^2}{n^2\tan^2(k\pi/n)}\right),$$

for some constant a. Multiplying out the terms in the formula (6.8), we see that $F_n(z) = z$ plus higher powers of z. This implies that $a = 1$ and completes the proof of the lemma. ☐

We are now ready to prove Euler's formula. First, by Lemma 6.9,

$$\sin z = \lim_{n\to\infty}\left\{z\prod_{k=1}^{m}\left(1 - \frac{z^2}{n^2\tan^2(k\pi/n)}\right)\right\} = \lim_{n\to\infty} z\prod_{k=1}^{\infty}\left(1 + a_k(n)\right),$$

where the limit is taken through *odd* natural numbers $n = 2m+1$, and $a_k(n) := -\frac{z^2}{n^2\tan^2(k\pi/n)}$ for $1 \le k \le m$ and $a_k(n) := 0$ else. Second, since $\lim_{z\to 0}\frac{\tan z}{z} = \lim_{z\to 0}\frac{\sin z}{z}\cdot\frac{1}{\cos z} = 1$, we see that

$$\lim_{n\to\infty} a_k(n) = \lim_{n\to\infty} -\frac{z^2}{n^2\tan^2(k\pi/n)} = \lim_{n\to\infty} -\frac{z^2}{\left(\frac{\tan(k\pi/n)}{1/n}\right)^2} = -\frac{z^2}{k^2\pi^2}.$$

Third, in Lemma 4.56 back in Section 4.10, we proved that

$$x < \tan x, \qquad \text{for } 0 < x < \pi/2. \tag{6.9}$$

Thus, for any complex number z, if $n = 2m+1$ and $1 \le k \le m$, then

$$|a_k(n)| = \left|\frac{z^2}{n^2\tan^2(k\pi/n)}\right| \le \frac{|z|^2}{n^2(k\pi)^2/n^2} = \frac{|z|^2}{k^2\pi^2}.$$

Finally, since the sum $\sum_{k=1}^\infty \frac{|z|^2}{k^2\pi^2}$ converges, by Tannery's theorem for infinite products, we have

$$\sin z = \lim_{n\to\infty} z\prod_{k=1}^\infty (1+a_k(n)) = z\prod_{k=1}^\infty \lim_{n\to\infty}(1+a_k(n)) = z\prod_{k=1}^\infty \left(1-\frac{z^2}{k^2\pi^2}\right).$$

After replacing z by πz, we get Euler's infinite product expansion for $\sin \pi z$. In particular, we see that

$$\pi i\prod_{k=1}^\infty\left(1+\frac{1}{k^2}\right) = \pi i\prod_{k=1}^\infty\left(1-\frac{i^2}{k^2}\right) = \sin \pi i = \frac{e^{-\pi}-e^{\pi}}{2i}.$$

Thus, we have derived the very pretty formula

$$\boxed{\frac{e^{\pi}-e^{-\pi}}{2\pi} = \prod_{n=1}^\infty\left(1+\frac{1}{n^2}\right).}$$

Recall from Section 6.1 how easy it was to find that $\prod_{n=1}^\infty\left(1-\frac{1}{n^2}\right) = 1/2$, but replacing $-1/n^2$ with $+1/n^2$ is a whole different story!

6.4.2. Expansion of sine II. Our second proof of Euler's infinite product for sine is based on the following neat identity involving sines instead of tangents!

Lemma 6.10. *If $n = 2m+1$ with $m \in \mathbb{N}$, then for any $z \in \mathbb{C}$,*

$$\sin nz = n\sin z\prod_{k=1}^m\left(1-\frac{\sin^2 z}{\sin^2(k\pi/n)}\right).$$

Proof. Lemma 2.26 shows that for any k, $2\cos kz$ is a polynomial in $2\cos z$ of degree k (with integer coefficients, although this fact is not important for this lemma). Technically speaking, Lemma 2.26 was proved under the assumption that z is real, but the proof only used the angle addition formula for cosine, which holds for complex variables as well. Any case, since $2\cos kz$ is a polynomial in $2\cos z$ of degree k, it follows that $\cos kz$ is a polynomial in $\cos z$ of degree k, say $\cos kz = Q_k(\cos z)$ where Q_k is a polynomial of degree k. In particular,

$$\cos 2kz = Q_k(\cos 2z) = Q_k(1-2\sin^2 z),$$

so $\cos 2kz$ is a polynomial of degree k in $\sin^2 z$. Now using the addition formulas for sine, we get

$$\sin(2k+1)z - \sin(2k-1)z = 2\sin z\cdot\cos(2kz) = 2\sin z\cdot Q_k(1-2\sin^2 z). \tag{6.10}$$

We claim that for any $m = 0,1,2,\ldots$, $\sin(2m+1)z$ is of the form

$$\sin(2m+1)z = \sin z\cdot P_m(\sin^2 z), \tag{6.11}$$

where P_m is a polynomial of degree m. For example, if $m = 0$, then $\sin z = \sin z\cdot P_0(\sin^2 z)$ where $P_0(w) = 1$ is the constant polynomial 1. If $m = 1$, then by (6.10), we have

$$\begin{aligned}\sin(3z) &= \sin z + 2\sin z\cdot Q_1(1-2\sin^2 z)\\ &= \sin z\cdot P_0(\sin^2 z) + 2\sin z\cdot Q_1(1-2\sin^2 z) = \sin z\cdot P_1(\sin^2 z),\end{aligned}$$

where $P_1(w) = P_0(w) + 2Q_1(1-2w)$. To prove (6.11) for general m just requires an induction argument based on (6.10), which we leave to the interested reader. Now, observe that $\sin(2m+1)z$ is zero when $z = z_k$ with $z_k = k\pi/(2m+1)$ where

$k = 1, 2, \ldots, m$. Also observe that since $0 < z_1 < z_2 < \cdots < z_m < \pi/2$, the m values $\sin z_k$ are distinct positive values. Hence, according to (6.11), $P_m(w) = 0$ at the m distinct values $w = \sin^2 z_k$, $k = 1, 2, \ldots, m$. Thus, as a consequence of the fundamental theorem of algebra, the polynomial $P_m(w)$ can be factored into a constant times

$$(w - z_1)(w - z_2) \cdots (w - z_m) = \prod_{k=1}^{m} \left(w - \sin^2 \left(\frac{k\pi}{2m+1} \right) \right) = \prod_{k=1}^{m} \left(w - \sin^2 \left(\frac{k\pi}{n} \right) \right),$$

(since $n = 2m + 1$) which is a constant times

$$\prod_{k=1}^{m} \left(1 - \frac{w}{\sin^2(k\pi/n)} \right).$$

Setting $w = \sin^2 z$, we obtain

$$\sin(2m+1)z = \sin z \cdot P_m(\sin^2 z) = a \sin z \cdot \prod_{k=1}^{m} \left(1 - \frac{\sin^2 z}{\sin^2(k\pi/n)} \right),$$

for some constant a. Since $\sin(2m+1)z/\sin z$ has limit equal to $2m+1$ as $z \to 0$, it follows that $a = 2m + 1$. This completes the proof of the lemma. □

For our second proof of Euler's infinite product for sine, we need to replace the identity (6.9) for tangents with a similar one involving sine.

Lemma 6.11. *There exists a constant $c > 0$ such that for $0 \le x \le \pi/2$,*

$$c\, x \le \sin x.$$

Also, if $|z| \le 1$, then

$$|\sin z| \le \frac{6}{5}|z|.$$

Proof. Since $\lim_{z\to 0} \frac{\sin z}{z} = 1$, the function $f(x) = \sin x/x$ is a continuous function of x in $[0, \pi/2]$ where we define $f(0) := 1$. Observe that f is positive on $[0, \pi/2]$ because $f(0) = 1 > 0$ and $\sin x > 0$ for $0 < x \le \pi/2$. Therefore, by the max/min value theorem, $f(x) \ge f(b) > 0$ on $[0, \pi/2]$ for some $b \in [0, \pi/2]$. This proves that $c\, x \le \sin x$ on $[0, \pi/2]$ where $c = f(b) > 0$.

To see that $|\sin z| \le \frac{6}{5}|z|$, observe that for $|z| \le 1$, we have $|z|^k \le |z|$ for any k, and

$$\begin{aligned}(2n+1)! &= (2 \cdot 3) \cdot (4 \cdot 5) \cdots (2n \cdot (2n+1)) \\ &\ge (2 \cdot 3) \cdot (2 \cdot 3) \cdots (2 \cdot 3) = (2 \cdot 3)^n = 6^n.\end{aligned}$$

Thus,

$$\begin{aligned}|\sin z| &= \left| \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \right| \le |z| + \frac{|z|^3}{3!} + \frac{|z|^5}{5!} + \cdots \\ &\le \left[1 + \frac{1}{3!} + \frac{1}{5!} + \cdots \right] |z| \le \left[1 + \frac{1}{6} + \frac{1}{6^2} + \cdots \right] |z| = \frac{6}{5}|z|.\end{aligned}$$

□

We are now ready to give our second proof of Euler's infinite product for sine. (This time we'll be brief since this is our second time through.) To this end, we let $n \geq 3$ be odd and we replace z by z/n in Lemma 6.10 to get

$$\sin z = n \sin(z/n) \prod_{k=1}^{m} \left(1 - \frac{\sin^2(z/n)}{\sin^2(k\pi/n)}\right),$$

where $n = 2m+1$. We now take $n \to \infty$ through odd integers. In doing so, we can always make sure that $n = 2m+1 > |z|$. In this case, according to Lemma 6.11,

$$\left|\frac{\sin^2(z/n)}{\sin^2(k\pi/n)}\right| \leq \frac{(6/5|z/n|)^2}{c^2(k\pi/n)^2} = C\frac{|z|^2}{k^2},$$

where $C = \frac{36}{25c^2\pi^2}$ is a constant. Since the sum $C|z|^2 \sum_{k=1}^{\infty} 1/k^2$ converges, and

$$\lim_{n\to\infty} n \sin(z/n) = \lim_{n\to\infty} \frac{\sin(z/n)}{1/n} = z,$$

and

$$\lim_{n\to\infty} \left(1 - \frac{\sin^2(z/n)}{\sin^2(k\pi/n)}\right) = \lim_{n\to\infty} \left(1 - \frac{\left(\frac{\sin(z/n)}{1/n}\right)^2}{\left(\frac{\sin(k\pi/n)}{1/n}\right)^2}\right) = \left(1 - \frac{z^2}{k^2\pi^2}\right),$$

Tannery's theorem for infinite products implies that

$$\sin z = \lim_{n\to\infty} \left\{ n \sin(z/n) \prod_{k=1}^{m} \left(1 - \frac{\sin^2(z/n)}{\sin^2(k\pi/n)}\right) \right\} = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2\pi^2}\right).$$

Finally, replacing z by πz completes the proof of Euler's formula.

6.4.3. Applications of Euler's product. For our first application, we derive the infinite product expansion for the cosine function. In fact, using the double angle formula for sine, we get

$$\cos \pi z = \frac{\sin 2\pi z}{2 \sin \pi z} = \frac{2\pi z \cdot \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{n^2}\right)}{2\pi z \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)} = \frac{\prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{n^2}\right)}{\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)}.$$

The top product can be split as a product of even and odd terms:

$$\prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right) \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n)^2}\right) = \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right) \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right),$$

from which we get

$$\boxed{\cos \pi z = \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right).}$$

For our second application, we derive John Wallis' (1616–1703) formulas for π.

COROLLARY 6.12 (**Wallis' formulas**). *We have*

$$\boxed{\begin{aligned}\frac{\pi}{2} &= \prod_{n=1}^{\infty} \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots , \\ \sqrt{\pi} &= \lim_{n\to\infty} \frac{1}{\sqrt{n}} \prod_{k=1}^{n} \frac{2k}{2k-1} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \cdot \frac{2}{1} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdots \frac{2n}{2n-1}.\end{aligned}}$$

PROOF. The first formula is obtained by setting $z = 1/2$ in Euler's infinite product expansion for sine and then taking reciprocals. To obtain the second formula, we write the first formula as

$$\frac{\pi}{2} = \lim_{n\to\infty} \left(\frac{2}{1}\right)^2 \cdot \left(\frac{4}{3}\right)^2 \cdots \left(\frac{2n}{2n-1}\right)^2 \cdot \frac{1}{2n+1},$$

so that

$$\sqrt{\pi} = \lim_{n\to\infty} \sqrt{\frac{2}{2n+1}} \prod_{k=1}^{n} \frac{2k}{2k-1} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \frac{1}{\sqrt{1+1/2n}} \prod_{k=1}^{n} \frac{2k}{2k-1}.$$

Using that $1/\sqrt{1+1/2n} \to 1$ as $n \to \infty$ completes our proof. □

We prove prove a beautiful expression for π due to Sondow [**164**] (which I found on Weisstein's website [**184**]). To present this formula, we first manipulate Wallis' first formula to

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2-1} = \prod_{n=1}^{\infty} \left(1 + \frac{1}{4n^2-1}\right).$$

Second, we observe that

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = \frac{1}{2} \cdot 1 = \frac{1}{2},$$

since the sum telescopes (see e.g. the telescoping series theorem — Theorem 3.24). Dividing these two formulas, we get

$$\boxed{\pi = \frac{\displaystyle\prod_{n=1}^{\infty} \left(1 + \frac{1}{4n^2-1}\right)}{\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2-1}},}$$

quite astonishing!

EXERCISES 6.4.

1. Put $z = \pi/4$ into the cosine expansion to derive the following elegant product for $\sqrt{2}$:

$$\boxed{\sqrt{2} = \left(1 + \frac{1}{1}\right)\left(1 - \frac{1}{3}\right)\left(1 + \frac{1}{5}\right)\left(1 - \frac{1}{7}\right)\left(1 + \frac{1}{9}\right) \cdots}.$$

2. Prove that

$$\sinh \pi z = \pi z \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{k^2}\right) \quad \text{and} \quad \cosh \pi z = \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{(2n-1)^2}\right).$$

3. In this problem we give three other proofs of the cosine expansion!

(a) Replace z by $-z+\pi/2$ in the sine product to derive the cosine product. Make sure that you justify any manipulations you make.
(b) For our second proof, show that for n even, we can write

$$\cos z = \prod_{k=1}^{n-1} \left(1 - \frac{\sin^2(z/n)}{\sin^2(k\pi/2n)}\right), \quad k = 1, 3, 5, \dots, n-1.$$

Using Tannery's theorem, deduce the cosine expansion.
(c) Write $\cos z = \lim_{n\to\infty} G_n(z)$, where

$$G_n(z) = \frac{1}{2}\left\{\left(1+\frac{iz}{n}\right)^n + \left(1-\frac{iz}{n}\right)^n\right\}.$$

Prove that if $n = 2m$ with $m \in \mathbb{N}$, then

$$G_n(z) = \prod_{k=0}^{m} \left(1 - \frac{z^2}{n^2\tan^2((2k+1)\pi/(2n))}\right).$$

Using Tannery's theorem, deduce the cosine expansion.

4. Prove the following splendid formula:

$$\boxed{\sqrt{\pi} = \lim_{n\to\infty} \frac{(n!)^2\, 2^{2n}}{(2n)!\,\sqrt{n}}.}$$

Suggestion: Wallis' formula is hidden here.

6.5. Partial fraction expansions of the trigonometric functions

The goal of this section is to prove Euler's partial fraction expansion (6.2):

THEOREM 6.13 (**Euler's partial fraction ($\frac{\pi}{\sin \pi z}$)**). *We have*

$$\boxed{\frac{\pi}{\sin \pi z} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{n^2 - z^2} \quad \textit{for all } z \in \mathbb{C}\setminus\mathbb{Z}.}$$

We also derive partial fraction expansions for the other trigonometric functions. We begin with the cotangent.

6.5.1. Partial fraction expansion of the cotangent.

We shall prove the following theorem (from which we'll derive the sine expansion).

THEOREM 6.14 (**Euler's partial fraction ($\pi z \cot \pi z$)**). *We have*

$$\boxed{\pi z \cot \pi z = 1 + 2z^2 \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2} \quad \textit{for all } z \in \mathbb{C}\setminus\mathbb{Z}.}$$

Our proof of Euler's expansion of the cotangent is based on the following lemma.

LEMMA 6.15. *For any noninteger complex number z and $n \in \mathbb{N}$, we have*

$$\pi z \cot \pi z = \frac{\pi z}{2^n}\cot\frac{\pi z}{2^n} + \sum_{k=1}^{2^{n-1}-1} \frac{\pi z}{2^n}\left(\cot\frac{\pi(z+k)}{2^n} + \cot\frac{\pi(z-k)}{2^n}\right) - \frac{\pi z}{2^n}\tan\frac{\pi z}{2^n}.$$

PROOF. Using the double angle formula

$$2\cot 2z = 2\frac{\cos 2z}{\sin 2z} = \frac{\cos^2 z - \sin^2 z}{\cos z \sin z} = \cot z - \tan z,$$

we see that

$$\cot 2z = \frac{1}{2}\Big(\cot z - \tan z\Big).$$

Replacing z with $\pi z/2$, we get

$$\cot \pi z = \frac{1}{2}\bigg(\cot\frac{\pi z}{2} - \tan\frac{\pi z}{2}\bigg). \tag{6.12}$$

Multiplying this equality by πz proves our lemma for $n = 1$. In order to proceed to induction, we note that since $\tan z = -\cot(z \pm \pi/2)$, we find that

$$\cot \pi z = \frac{1}{2}\bigg(\cot\frac{\pi z}{2} + \cot\frac{\pi(z \pm 1)}{2}\bigg). \tag{6.13}$$

This is the main formula on which induction may be applied to prove our lemma. For instance, let's take the case $n = 2$. Considering the positive sign in the second cotangent, we have

$$\cot \pi z = \frac{1}{2}\bigg(\cot\frac{\pi z}{2} + \cot\frac{\pi(z+1)}{2}\bigg).$$

Applying (6.13) to each cotangent on the right of this equation, using the plus sign for the first and the minus sign for the second, we get

$$\begin{aligned}\cot \pi z &= \frac{1}{2^2}\left\{\bigg(\cot\frac{\pi z}{2^2} + \cot\frac{\pi(\frac{z}{2}+1)}{2}\bigg) + \bigg(\cot\frac{\pi(z+1)}{2^2} + \cot\frac{\pi(\frac{z+1}{2}-1)}{2}\bigg)\right\}\\ &= \frac{1}{2^2}\left\{\cot\frac{\pi z}{2^2} + \cot\frac{\pi(z+2)}{2^2} + \cot\frac{\pi(z+1)}{2^2} + \cot\frac{\pi(z-1)}{2^2}\right\},\end{aligned}$$

which, after bringing the second cotangent to the end, takes the form

$$\cot \pi z = \frac{1}{2^2}\left\{\cot\frac{\pi z}{2^2} + \cot\frac{\pi(z+1)}{2^2} + \cot\frac{\pi(z-1)}{2^2} + \cot\Big(\frac{\pi z}{2^2} + \frac{\pi}{2}\Big)\right\}.$$

However, the last term is exactly $-\tan \pi z/2^2$, and so our lemma is proved for $n = 2$. Continuing by induction proves our lemma for general n. □

Fix a noninteger z; we shall prove Euler's expansion for the cotangent. Since

$$\lim_{n\to\infty} \frac{\pi z}{2^n}\cot\frac{\pi z}{2^n} = \lim_{w\to 0} w\cot w = \lim_{w\to 0}\frac{w}{\sin w}\cdot\cos w = 1, \tag{6.14}$$

and $\lim_{n\to\infty}\frac{\pi z}{2^n}\tan(\frac{\pi z}{2^n}) = 0\cdot 0 = 0$, taking $n \to \infty$ in the formula from the preceding lemma, we conclude that

$$\pi z \cot \pi z = 1 + \lim_{n\to\infty}\left\{\sum_{k=1}^{2^{n-1}-1}\frac{\pi z}{2^n}\bigg(\cot\frac{\pi(z+k)}{2^n} + \cot\frac{\pi(z-k)}{2^n}\bigg)\right\}. \tag{6.15}$$

We shall apply Tannery's theorem to this sum. To bound each term in the sum, we use the formula

$$\cot(\alpha+\beta) + \cot(\alpha-\beta) = \frac{\sin 2\alpha}{\sin^2\alpha - \sin^2\beta}.$$

This formula is obtained by expressing $\cot(\alpha \pm \beta)$ in terms of cosine and sine and using the angle addition formulas for these functions (the diligent reader will supply the details!). Setting $\alpha = \pi z/2^n$ and $\beta = \pi k/2^n$, we obtain

$$\cot\frac{\pi(z+k)}{2^n} + \cot\frac{\pi(z-k)}{2^n} = \frac{\sin 2\alpha}{\sin^2\alpha - \sin^2\beta},$$

where we keep the notation $\alpha = \pi z/2^n$ and $\beta = \pi k/2^n$ on the right. Now according to Lemma 6.11, for n large so that $|\alpha| = |\pi z/2^n|$ is less than $1/2$, we have

$$|\sin 2\alpha| \le \frac{6}{5}|2\alpha| \le 3|\alpha| \quad \text{and} \quad |\sin\alpha| \le \frac{6}{5}|\alpha| \le 2|\alpha|,$$

and, since $\beta = \pi k/2^n < \pi/2$ for $k = 1, \ldots, 2^{n-1} - 1$, for some $c > 0$,

$$c\,\beta \le \sin\beta.$$

Hence,

$$c^2\,\beta^2 \le \sin^2\beta \le |\sin^2\alpha - \sin^2\beta| + |\sin^2\alpha| \le |\sin^2\alpha - \sin^2\beta| + 4|\alpha|^2$$
$$\Longrightarrow \quad c^2\,\beta^2 - 4|\alpha|^2 \le |\sin^2\alpha - \sin^2\beta|.$$

Choose k such that $c\,k > 2|z|$. Then

$$c^2\,\beta^2 = c^2\left(\frac{\pi k}{2^n}\right)^2 = \left(\frac{\pi c\,k}{2^n}\right)^2 > 4\left(\frac{\pi|z|}{2^n}\right)^2 = 4|\alpha|^2 \quad \Longrightarrow \quad c^2\,\beta^2 - 4|\alpha|^2 > 0,$$

and combining this with the preceding line, we obtain

$$0 < c^2\,\beta^2 - 4|\alpha|^2 \le |\sin^2\alpha - \sin^2\beta|.$$

Hence,

$$\frac{|\sin 2\alpha|}{|\sin^2\alpha - \sin^2\beta|} \le \frac{3|\alpha|}{c^2\,\beta^2 - 4|\alpha|^2} = \frac{3\pi|z|/2^n}{c^2(\pi k/2^n)^2 - 4(\pi|z|/2^n)^2} = 3\frac{2^n|z|/\pi}{c^2\,k^2 - 4|z|^2}.$$

Thus, for $c\,k > 2|z|$, we have

$$\left|\frac{\pi z}{2^n}\left(\cot\frac{\pi(z+k)}{2^n} + \cot\frac{\pi(z-k)}{2^n}\right)\right| \le \frac{3|z|^2}{c^2\,k^2 - 4|z|^2}.$$

Observe that the sum

$$\sum_k \frac{3|z|^2}{c^2\,k^2 - 4|z|^2},$$

starting from $k > 2|z|/c$, is a convergent series of positive numbers. Now using that $\lim_{z\to 0} z\cot z = 1$ from (6.14), we see that

$$\lim_{n\to\infty}\frac{\pi z}{2^n}\cot\frac{\pi(z+k)}{2^n} = \lim_{n\to\infty}\frac{\frac{\pi z}{2^n}}{\frac{\pi(z+k)}{2^n}}\cdot\frac{\pi(z+k)}{2^n}\cot\frac{\pi(z+k)}{2^n} = \frac{z}{z+k},$$

and in a similar way,

$$\lim_{n\to\infty}\frac{\pi z}{2^n}\cot\frac{\pi(z-k)}{2^n} = \frac{z}{z-k}.$$

Thus,

$$\lim_{n\to\infty}\frac{\pi z}{2^n}\left(\cot\frac{\pi(z+k)}{2^n} + \cot\frac{\pi(z-k)}{2^n}\right) = \frac{z}{z+k} + \frac{z}{z-k} = \frac{2z^2}{z^2-k^2}.$$

Finally, Tannery's theorem applied to the sum (6.15) gives

$$\pi z\cot\pi z = 1 + 2z^2\sum_{k=1}^{\infty}\frac{1}{z^2-k^2},$$

which proves Euler's cotangent expansion.

6.5.2. Partial fraction expansions of the other trig functions. We shall leave most of the details to the exercises. Using the formula (see (6.12))

$$\pi \tan \frac{\pi z}{2} = \pi \cot \frac{\pi z}{2} - 2\pi \cot \pi z,$$

and substituting in the partial fraction expansion of the cotangent, gives, as the diligent reader will do in Problem 1, for noninteger $z \in \mathbb{C}$,

$$\boxed{\pi \tan \frac{\pi z}{2} = \sum_{n=0}^{\infty} \frac{4z}{(4n+1)^2 - z^2}.} \tag{6.16}$$

To derive a partial fraction expansion for $\frac{\pi}{\sin \pi z}$, we first derive the identity

$$\frac{1}{\sin z} = \cot z + \tan \frac{z}{2}.$$

To see this, observe that

$$\begin{aligned}\cot z + \tan \frac{z}{2} = \frac{\cos z}{\sin z} + \frac{\sin(z/2)}{\cos(z/2)} &= \frac{\cos z\, \cos(z/2) + \sin z\, \sin(z/2)}{\sin z\, \cos(z/2)} \\ &= \frac{\cos(z - (z/2))}{\sin z\, \cos(z/2)} = \frac{\cos(z/2)}{\sin z\, \cos(z/2)} = \frac{1}{\sin z}.\end{aligned}$$

This identity, together with the partial fraction expansions of the tangent and cotangent and a little algebra, which the extremely diligent reader will supply in Problem 1, imply that for noninteger $z \in \mathbb{C}$,

$$\boxed{\frac{\pi}{\sin \pi z} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{n^2 - z^2}.} \tag{6.17}$$

Finally, the incredibly awesome diligent reader ☺ will supply the details for the following cosine expansion: For noninteger $z \in \mathbb{C}$,

$$\boxed{\frac{\pi}{4\cos \frac{\pi z}{2}} = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)}{(2n+1)^2 - z^2}.} \tag{6.18}$$

Exercises 6.5.

1. Fill in the details for the proofs of (6.16) and (6.17). For (6.18), first show that
$$\frac{\pi}{\sin \pi z} = \frac{1}{z} + \Big(\frac{1}{1-z} - \frac{1}{1+z}\Big) - \Big(\frac{1}{2-z} - \frac{1}{2+z}\Big) + \cdots .$$
Replacing z with $\frac{1-z}{2}$ and doing some algebra, derive the expansion (6.18).
2. Derive Gregory-Leibniz-Madhava's series $\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$ by replacing $z = 1/4$ in the partial fraction expansions of $\pi z \cot \pi z$ and $\pi / \sin \pi z$. How can you derive Gregory-Leibniz-Madhava's series from the expansion of $\frac{\pi}{4\cos\frac{\pi z}{2}}$?
3. Derive the following formulas for π:
$$\pi = z \tan\Big(\frac{\pi}{z}\Big) \cdot \Big[1 - \frac{1}{z-1} + \frac{1}{z+1} - \frac{1}{2z-1} + \frac{1}{2z+1} - + \cdots\Big]$$
and
$$\pi = z \sin\Big(\frac{\pi}{z}\Big) \cdot \Big[1 + \frac{1}{z-1} - \frac{1}{z+1} - \frac{1}{2z-1} + \frac{1}{2z+1} + - - + + \cdots\Big].$$
In particular, plug in $z = 3, 4, 6$ to derive some pretty formulas.

6.6. ★ More proofs that $\pi^2/6 = \sum_{n=1}^\infty 1/n^2$

In this section, we continue our discussion from Section 5.11 concerning the Basel problem of determining the sum of the reciprocals of the squares. In this section we (basically) present Euler's original proof but with all the details he left out! A good reference for this material is [**84**] and for more on Euler, see [**8**].

6.6.1. Proof IV: (Basically) Euler's original proof! (Cf. [**37**, p. 74].) We begin with Euler's sine expansion restricted to $0 \le x < 1$:

$$\frac{\sin \pi x}{\pi x} = \prod_{n=1}^\infty \Big(1 - \frac{x^2}{n^2}\Big).$$

By Theorem 6.3, we have $\sin \pi x/\pi x = e^{L(x)}$, where

$$L(x) = \sum_{n=1}^\infty \log \Big(1 - \frac{x^2}{n^2}\Big).$$

Taking logs of both sides of the equation $\sin \pi x/\pi x = e^{L(x)}$, we obtain

$$\log\left(\frac{\sin \pi x}{\pi x}\right) = \sum_{n=1}^\infty \log\Big(1 - \frac{x^2}{n^2}\Big), \quad 0 \le x < 1.$$

Replacing x by $-x^2/n^2$ in the infinite series representation

$$\log(1+x) = \sum_{m=1}^\infty \frac{(-1)^{m-1}}{m} x^m,$$

we get

$$\log\left(\frac{\sin \pi x}{\pi x}\right) = -\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{1}{m}\frac{x^{2m}}{n^{2m}}, \quad 0 \le x < 1.$$

Since

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \left|\frac{1}{m}\frac{x^{2m}}{n^{2m}}\right| = \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{1}{m}\frac{|x|^{2m}}{n^{2m}} = \sum_{n=1}^\infty \log\Big(1 - \frac{|x|^2}{n^2}\Big) = \log\left(\frac{\sin \pi |x|}{\pi |x|}\right) < \infty,$$

by Cauchy's double series theorem, we can iterate sums:

$$\begin{aligned}\log\left(\frac{\sin \pi x}{\pi x}\right) &= -\sum_{m=1}^\infty \left(\sum_{n=1}^\infty \frac{1}{n^{2m}}\right)\frac{x^{2m}}{m} \\ &= x^2 \sum_{n=1}^\infty \frac{1}{n^2} + \frac{x^4}{2}\sum_{n=1}^\infty \frac{1}{n^4} + \frac{x^6}{3}\sum_{n=1}^\infty \frac{1}{n^6} + \cdots .\end{aligned} \tag{6.19}$$

On the other hand, by our power series composition theorem, we have (after some simplification)

$$\begin{aligned}-\log\left(\frac{\sin \pi x}{\pi x}\right) &= -\log\left(1 - \left(\frac{\pi^2 x^2}{3!} - \frac{\pi^4 x^4}{5!} + - \cdots\right)\right) \\ &= \left(\frac{\pi^2 x^2}{3!} - \frac{\pi^4 x^4}{5!} + - \cdots\right) + \frac{1}{2}\left(\frac{\pi^2 x^2}{3!} - \frac{\pi^4 x^4}{5!} + - \cdots\right)^2 + \cdots \\ &= \frac{\pi^2}{3!}x^2 + \left(-\frac{\pi^4}{5!} + \frac{\pi^4}{2\cdot(3!)^2}\right)x^4 + \left(\frac{\pi^6}{7!} - \frac{\pi^6}{3!\cdot 5!} + \frac{\pi^6}{3\cdot(3!)^3}\right)x^6 + \cdots .\end{aligned} \tag{6.20}$$

Equating this with (6.19), we obtain

$$\frac{\pi^2}{3!}x^2 + \left(-\frac{\pi^4}{5!} + \frac{\pi^4}{2\cdot(3!)^2}\right)x^4 + \left(\frac{\pi^6}{7!} - \frac{\pi^6}{3!\cdot 5!} + \frac{\pi^6}{3\cdot(3!)^3}\right)x^6 + \cdots$$
$$= x^2\sum_{n=1}^\infty \frac{1}{n^2} + \frac{x^4}{2}\sum_{n=1}^\infty \frac{1}{n^4} + \frac{x^6}{3}\sum_{n=1}^\infty \frac{1}{n^6} + \cdots,$$

or after simplification,

$$\frac{\pi^2}{6}x^2 + \frac{\pi^4}{180}x^4 + \frac{\pi^6}{2835}x^6 + \cdots = x^2\sum_{n=1}^\infty \frac{1}{n^2} + \frac{x^4}{2}\sum_{n=1}^\infty \frac{1}{n^4} + \frac{x^6}{3}\sum_{n=1}^\infty \frac{1}{n^6} + \cdots. \tag{6.21}$$

By the identity theorem, the coefficients of x must be identical. Thus, comparing the x^2 terms, we get Euler's formula:

$$\frac{\pi^2}{6} = \sum_{n=1}^\infty \frac{1}{n^2},$$

comparing the x^4 terms, we get

$$\boxed{\frac{\pi^4}{90} = \sum_{n=1}^\infty \frac{1}{n^4},} \tag{6.22}$$

and finally, comparing the x^6 terms, we get

$$\boxed{\frac{\pi^6}{945} = \sum_{n=1}^\infty \frac{1}{n^6}.} \tag{6.23}$$

Now what if we took more terms in (6.19) and (6.20), say to x^{2k}, can we then find a formula for $\sum 1/n^{2k}$? The answer is certainly true but the work required to get a formula is rather intimidating; see Problem 1 for a formula when $k = 4$. In Section 6.8 we find formulas for $\zeta(2k)$ for *all* k in terms of π and the Bernoulli numbers!

6.6.2. Proof V. (Cf. [**94**], [**39**].) For this proof, we start with Lemma 6.10, which states that if $n = 2m + 1$ with $m \in \mathbb{N}$, then

$$\sin nz = n\sin z\prod_{k=1}^m \left(1 - \frac{\sin^2 z}{\sin^2(k\pi/n)}\right). \tag{6.24}$$

We fix an m; later we shall take $m \to \infty$. Substituting in the expansion

$$\sin nz = nz - \frac{n^3z^3}{3!} + \frac{n^5z^5}{5!} - + \cdots$$

into the left-hand side of (6.24), and the expansions

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots,$$

and

$$\sin^2 z = \frac{1}{2}(1 - \cos 2z) = z^2 - \frac{2}{3}z^4 + - \cdots,$$

into the right-hand side of (6.24), and then multiplying everything out and simplifying, we obtain

$$nz - \frac{n^3 z^3}{3!} + - \cdots = nz + \left(-\frac{n}{6} - n\sum_{k=1}^{m} \frac{1}{\sin^2(k\pi/n)} \right) z^3 + \cdots .$$

Comparing the z^3 terms, by the identity theorem we conclude that

$$-\frac{n^3}{6} = -\frac{n}{6} - n\sum_{k=1}^{m} \frac{1}{\sin^2(k\pi/n)},$$

or

$$\frac{1}{6} - \sum_{k=1}^{m} \frac{1}{n^2\sin^2(k\pi/n)} = \frac{1}{n^2}. \tag{6.25}$$

To establish Euler's formula, we apply Tannery's theorem to this sum. According to Lemma 6.11, for some positive constant c,

$$c\,x \le \sin x \qquad \text{for } 0 \le x \le \pi/2. \tag{6.26}$$

Now for $0 \le k \le m = (n-1)/2$, we have $k\pi/n < \pi/2$, so for such k,

$$c \cdot \frac{k\pi}{n} \le \sin \frac{k\pi}{n},$$

which gives

$$\frac{1}{n^2} \cdot \frac{1}{\sin^2(k\pi/n)} \le \frac{1}{n^2} \cdot \frac{n^2}{(c\pi)^2 k^2} = \frac{1}{c^2\pi^2} \cdot \frac{1}{k^2}.$$

By the p-test, we know that the sum

$$\sum_{k=1}^{\infty} \frac{1}{c^2\pi^2} \cdot \frac{1}{k^2}$$

converges. Also, since $n\sin(x/n) \to x$ as $n \to \infty$, which implies that

$$\lim_{n\to\infty} \frac{1}{n^2\sin^2(k\pi/n)} = \frac{1}{k^2\pi^2},$$

taking $m \to \infty$ in (6.25), Tannery's theorem gives

$$\frac{1}{6} - \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2} = 0,$$

which is equivalent to Euler's formula. See Problem 2 for a proof that uses (6.25) but doesn't use Tannery's theorem.

6.6.3. Proof VI. In this proof, we follow Hofbauer [**79**]. Back in Section 5.11, we derived the identity (see (5.81))

$$1 = \frac{2}{4^n} \sum_{k=0}^{2^{n-1}-1} \frac{1}{\sin^2 \frac{(2k+1)\pi}{2^{n+1}}}. \tag{6.27}$$

In Section 5.11, we established Euler's formula from this identity using Lemma 4.56. This time we apply Tannery's theorem. To do so, observe that by Lemma 6.11 (that is, the inequality (6.26)), for $0 \le k \le 2^{n-1} - 1$ we have

$$c \cdot \frac{(2k+1)\pi}{2^{n+1}} \le \sin \frac{(2k+1)\pi}{2^{n+1}}.$$

This implies that

$$\frac{2}{4^n} \cdot \frac{1}{\sin^2 \frac{(2k+1)\pi}{2^{n+1}}} \le \frac{2}{4^n} \cdot \frac{4^{n+1}}{(c\pi)^2 (2k+1)^2} = \frac{8}{c^2\pi^2} \cdot \frac{1}{(2k+1)^2}.$$

Since the sum

$$\sum_{k=0}^{\infty} \frac{8}{c^2\pi^2} \cdot \frac{1}{(2k+1)^2}$$

converges, and

$$\lim_{n\to\infty} \frac{2}{4^n} \cdot \frac{1}{\sin^2 \frac{(2k+1)\pi}{2^{n+1}}} = \lim_{n\to\infty} 8 \cdot \frac{1}{\left(2^{n+1} \sin \frac{(2k+1)\pi}{2^{n+1}}\right)^2} = \frac{8}{\pi^2 (2k+1)^2},$$

taking $n \to \infty$ in (6.27) and invoking Tannery's theorem, we obtain

$$1 = \sum_{k=0}^{\infty} \frac{8}{\pi^2 (2k+1)^2} = \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \quad \Longrightarrow \quad \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

From Section 5.11 (see the work around (5.79)) we know that this formula implies Euler's formula.

Exercises 6.6.

1. Determine the sum $\sum_{n=1}^{\infty} \frac{1}{n^8}$ using Euler's method; that is, in the same manner as we derived (6.22) and (6.23).
2. (Cf. [**39**]) (**Euler's sum, Proof VII**) Instead of using Tannery's theorem to derive Euler's formula from (6.25), we can follow Kortram [**94**] as follows.
   (i) Fix any $M \in \mathbb{N}$ and let $m > M$. Using (6.25), prove that for $n = 2m+1$,
   $$\frac{1}{6} - \sum_{k=1}^{M} \frac{1}{n^2 \sin^2(k\pi/n)} = \frac{1}{n^2} + \sum_{k=M+1}^{m} \frac{1}{n^2 \sin^2(k\pi/n)}.$$
   (ii) Using that $cx \le \sin x$ for $0 \le x \le \pi/2$ with $c > 0$, prove that
   $$0 \le \frac{1}{6} - \sum_{k=1}^{M} \frac{1}{n^2 \sin^2(k\pi/n)} \le \frac{1}{n^2} + \frac{1}{c^2\pi^2} \sum_{k=M+1}^{\infty} \frac{1}{k^2}.$$
   (iii) Finally, letting $m \to \infty$ (so that $n = 2m+1 \to \infty$ as well) and then letting $M \to \infty$, establish Euler's formula.
3. (**Partial fraction expansion of $1/\sin^2 x$, Proof III**) Recall from Problem 3 in Exercises 5.11 that for any $n \in \mathbb{N}$,
   $$\frac{1}{\sin^2 x} = \frac{1}{2^{2n}} \sum_{k=-2^{n-1}}^{2^{n-1}-1} \frac{1}{\sin^2 \frac{x+\pi k}{2^n}}.$$
   Show that you can apply Tannery's theorem to the right-hand series to derive the formula
   $$\boxed{\frac{1}{\sin^2 x} = \sum_{k\in\mathbb{Z}} \frac{1}{(x+\pi k)^2}.}$$
4. Following Hofbauer [**79**], we present an elementary proof of Gregory-Leibniz-Madhava's formula which is similar to the proofs found in this section for $\pi^2/6$.
   (i) Using the identity (6.13), prove that
   $$\cot \pi z = \frac{1}{2}\left(\cot \frac{\pi z}{2} - \cot \frac{\pi(1-z)}{2}\right).$$

(ii) Use the formula in (i) and induction to prove that for any $n \in \mathbb{N}$,

$$1 = \frac{1}{2^n} \sum_{k=0}^{2^{n-1}-1} \left(\cot \frac{(4k+1)\pi}{4 \cdot 2^n} - \cot \frac{(4k+3)\pi}{4 \cdot 2^n} \right).$$

(iii) Prove that $\cot z - \cot w = \sin(w - z)/(\sin z \, \sin w)$, and then prove that

$$1 = \frac{1}{2^n} \sum_{k=0}^{2^{n-1}-1} \frac{\sin \frac{\pi}{2^{n+1}}}{\sin \frac{(4k+1)\pi}{4 \cdot 2^n} \cdot \sin \frac{(4k+3)\pi}{4 \cdot 2^n}}.$$

(iv) Show that Tannery's theorem can be applied to the sum in (iii) as $n \to \infty$ and derive the equation

$$\frac{\pi}{8} = \sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+3)}.$$

Finally, show that this sum is equivalent to Gregory-Leibniz-Madhava's sum.

6.7. Riemann's remarkable ζ-function (bonus: probability and $\pi^2/6$)

We have already seen the Riemann zeta function at work in many examples. In this section we're going to look at some of its relations with number theory; this will give just a hint as to its great importance in mathematics ... we'll take up this discussion once more in Chapter 12 when we state (but not solve!) perhaps the most celebrated unsolved problem in mathematics: *The Riemann hypothesis.* As a consolation prize to our discussion on Riemann's ζ-function we'll find an incredible connection between probability theory and $\pi^2/6$.

6.7.1. The Riemann-zeta function and number theory. Our first relation is the following result proved by Euler which connects $\zeta(z)$ to prime numbers. The following proof uses Cauchy's multiplication theorem rather strongly; see Problem 1 for a proof using the good ole (rather elementary) Tannery's theorem!

THEOREM 6.16 (**Euler and Riemann**). *For all $z \in \mathbb{C}$ with $\operatorname{Re} z > 1$, we have*

$$\boxed{\zeta(z) = \prod \left(1 - \frac{1}{p^z}\right)^{-1} = \prod \frac{p^z}{p^z - 1},}$$

where the infinite product is over all prime numbers $p \in \mathbb{N}$.

PROOF. Let $r > 1$ be arbitrary and let $\operatorname{Re} z \geq r$. Let $2 < N \in \mathbb{N}$ and let $2 < 3 < \cdots < m < N$ be all the primes less than N. Then for every natural $n < N$, by unique factorization,

$$n^z = \left(2^i \, 3^j \, \cdots \, m^k\right)^z = 2^{iz} \, 3^{jz} \, \cdots \, m^{kz}$$

for some nonnegative integers $i, j, \ldots, k$. Using this fact, it follows that the product

$$\begin{aligned}\prod_{p<N} \left(1 - \frac{1}{p^z}\right)^{-1} &= \left(1 - \frac{1}{2^z}\right)^{-1} \left(1 - \frac{1}{3^z}\right)^{-1} \cdots \left(1 - \frac{1}{m^z}\right)^{-1} \\ &= \left(1 + \frac{1}{2^z} + \frac{1}{2^{2z}} + \frac{1}{2^{3z}} \cdots \right) \left(1 + \frac{1}{3^z} + \frac{1}{3^{2z}} + \frac{1}{3^{3z}} + \cdots \right) \cdots \\ &\qquad \cdots \left(1 + \frac{1}{m^z} + \frac{1}{m^{2z}} + \frac{1}{m^{3z}} + \cdots \right),\end{aligned}$$

after multiplying out and using Cauchy's multiplication theorem (or rather its generalization to a product of more than two absolutely convergent series), contains the numbers $1, \frac{1}{2^z}, \frac{1}{3^z}, \frac{1}{4^z}, \frac{1}{5^z} \ldots, \frac{1}{(N-1)^z}$ (along with all other numbers $\frac{1}{n^z}$ with $n \geq N$ having prime factors $2, 3, \ldots, m$). In particular,

$$\Big|\sum_{n=1}^{\infty} \frac{1}{n^z} - \prod_{p<N} \Big(1 - \frac{1}{p^z}\Big)^{-1}\Big| \leq \sum_{n=N}^{\infty} \Big|\frac{1}{n^z}\Big| \leq \sum_{n=N}^{\infty} \frac{1}{n^r},$$

since $\operatorname{Re} z \geq r$. By the p-test (with $p = r > 1$), $\sum \frac{1}{n^r}$ converges so the right-hand side tends to zero as $N \to \infty$. This completes our proof. □

In particular, since we know that $\zeta(2) = \pi^2/6$, we have

$$\boxed{\frac{\pi^2}{6} = \prod \frac{p^2}{p^2 - 1} = \frac{2^2}{2^2 - 1} \cdot \frac{3^2}{3^2 - 1} \cdot \frac{5^2}{5^2 - 1} \cdots}$$

Our next connection is with the following strange (but interesting) function:

$$\mu(n) := \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n = p_1\, p_2\, \cdots p_k \text{ is a product } k \text{ distinct prime numbers} \\ 0 & \text{else.} \end{cases}$$

This function is called the **Möbius function** after August Ferdinand Möbius (1790–1868). Some of its values are

$$\mu(1) = 1\ ,\ \mu(2) = -1\ ,\ \mu(3) = -1\ ,\ \mu(4) = 0\ ,\ \mu(5) = -1\ ,\ \mu(6) = 1\ ,\ \ldots .$$

THEOREM 6.17. *For all $z \in \mathbb{C}$ with $\operatorname{Re} z > 1$, we have*

$$\boxed{\frac{1}{\zeta(z)} = \prod \Big(1 - \frac{1}{p^z}\Big) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^z}.}$$

PROOF. Let $r > 1$ be arbitrary and let $\operatorname{Re} z \geq r$. Let $2 < N \in \mathbb{N}$ and let $2 < 3 < \cdots < m < N$ be all the primes less than N. Then observe that the product

$$\prod_{n<N} \Big(1 - \frac{1}{p^z}\Big) = \Big(1 + \frac{-1}{2^z}\Big)\Big(1 + \frac{-1}{3^z}\Big)\Big(1 + \frac{-1}{5^z}\Big) \cdots \Big(1 + \frac{-1}{m^z}\Big),$$

when multiplied out contains 1 and all numbers of the form

$$\Big(\frac{-1}{p_1^z}\Big) \cdot \Big(\frac{-1}{p_2^z}\Big) \cdot \Big(\frac{-1}{p_3^z}\Big) \cdots \Big(\frac{-1}{p_k^z}\Big) = \frac{(-1)^k}{p_1^z p_2^z \cdots p_k^z} = \frac{(-1)^k}{n^z}, \qquad n = p_1\, p_2\, \ldots\, p_k,$$

where $p_1 < p_2 < \cdots < p_k < N$ are distinct primes. In particular, $\prod_{n<N} \Big(1 - \frac{1}{p^z}\Big)$ contains the numbers $\frac{\mu(n)}{n^z}$ for $n = 1, 2, \ldots, N-1$ (along with all other numbers $\frac{\mu(n)}{n^z}$ with $n \geq N$ having prime factors $2, 3, \ldots, m$), so

$$\Big|\sum_{n=1}^{\infty} \frac{\mu(n)}{n^z} - \prod_{p<N} \Big(1 - \frac{1}{p^z}\Big)\Big| \leq \sum_{n=N}^{\infty} \Big|\frac{\mu(n)}{n^z}\Big| \leq \sum_{n=N}^{\infty} \frac{1}{n^r},$$

since $\operatorname{Re} z \geq r$. By the p-test (with $p = r > 1$), $\sum \frac{1}{n^r}$ converges so the right-hand side tends to zero as $N \to \infty$. This completes our proof. □

See the exercises for other neat connections of $\zeta(z)$ with number theory.

6.7.2. The eta function. A function related to the zeta function is the "alternating zeta function" or **Dirichlet eta-function**:

$$\eta(z) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}.$$

We can write the eta function in terms of the zeta function as follows.

THEOREM 6.18. *We have*

$$\eta(z) = (1 - 2^{1-z})\zeta(z), \quad z > 1.$$

PROOF. Splitting into sums of even and odd numbers, we get

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z} &= -\sum_{n=1}^{\infty} \frac{1}{(2n)^z} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^z} \\ &= -\sum_{n=1}^{\infty} \frac{1}{2^z}\frac{1}{n^z} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^z} \\ &= -2^{-z}\zeta(z) + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^z}.\end{aligned}$$

On the other hand, breaking the zeta function into sums of even and odd numbers, we get

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} \frac{1}{(2n)^z} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^z} = 2^{-z}\zeta(z) + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^z}.$$

Substituting this expression into the previous one, we see that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z} = -2^{-z}\zeta(z) + \zeta(z) - 2^{-z}\zeta(z),$$

which is the expression that we desired to prove. □

We now consider an incredible and shocking connection between probability theory, prime numbers, divisibility, and $\pi^2/6$.[1] Consider the following question: What is the probability that a natural number, chosen at random, is square free? Answer (drum role please): $6/\pi^2$. Here's another question: What is the probability that two given numbers, chosen at random, are relatively prime? Answer (drum role please): $6/\pi^2$.

6.7.3. Elementary probability theory. To prove these results with complete rigor would force us to devote a whole chapter to probability, so to avoid these details we'll assume some basic probability that should be "obvious" (or at least believable) to you; see [**2**] for a book on probability in case you want the hardcore theory. We only need the basics. We denote the probability, or chance, that an event A happens by $P(A)$. The classic definition is

$$P(A) = \frac{\text{number of occurrences of } A}{\text{total number of possibilities}}. \tag{6.28}$$

[1] *Such shocking connections in science perhaps made Albert Einstein (1879–1955) state that "the scientist's religious feeling takes the form of a rapturous amazement at the harmony of natural law, which reveals an intelligence of such superiority that, compared with it, all the systematic thinking and acting of human beings is an utterly insignificant reflection".* [**80**]

For example, consider a classroom with 10 people, m men and w women (so that $m + w = 10$). The probability of randomly "choosing a man" ($= M$) is

$$P(M) = \frac{\text{number of men}}{\text{total number of possibilities}} = \frac{m}{10}.$$

Similarly, the probability of randomly choosing a woman is $w/10$. We next need to discuss complementary events. If A^c is the event that A does not happen, then

$$P(A^c) = 1 - P(A). \tag{6.29}$$

For instance, according to (6.29) the probability of "*not* choosing a man", M^c, should be $P(M^c) = 1 - P(M) = 1 - m/10$. But this is certainly true because "*not* choosing a man" is the same as "choosing a woman" W, so recalling that $m + w = 10$, we have

$$P(M^c) = P(W) = \frac{w}{10} = \frac{10 - m}{10} = 1 - \frac{m}{10}.$$

Finally, we need to discuss independence. Whenever an event A is *unrelated* to an event B (such events are called **independent**), we have the fundamental relation:

$$P(A \text{ and } B) = P(A) \cdot P(B).$$

For example, let's say that we have two classrooms of 10 students each, the first one with m_1 men and w_1 women, and the second one with m_2 men and w_2 women. Let us randomly choose a pair of students, one from the first classroom and the other from the second. What is the probability of randomly "choosing a man from the first classroom" $= A$ *and* randomly "choosing a woman from the second classroom" $= B$? Certainly A and B don't depend on each other, so by our formula above we should have

$$P(A \text{ and } B) = P(A) \cdot P(B) = \frac{m_1}{10} \cdot \frac{w_2}{10} = \frac{m_1 w_2}{100}.$$

To see that this is indeed true, note that the number of ways to pair a man in classroom 1 with a woman in classroom 2 is $m_1 \cdot w_2$ and the total number of possible pairs of people is $10^2 = 100$. Thus,

$$P(A \text{ and } B) = \frac{\text{number of men-women pairs}}{\text{total number of possible pairs of people}} = \frac{m_1 \cdot m_2}{100},$$

in agreement with our previous calculation. We remark that for any number of events $A_1, A_2, \ldots$, which are unrelated to each other, we have the generalized result:

$$P(A_1 \text{ and } A_2 \text{ and } \cdots) = P(A_1) \cdot P(A_2) \cdots . \tag{6.30}$$

6.7.4. Probability and $\pi^2/6$. To begin discussing our two incredible and shocking problems, we first look at the following question: Given a natural number k, what is the probability, or chance, that a randomly chosen natural number is divisible by k? Since the definition (6.28) involves finite quantities, we can't use this definition as it stands. We can instead use the following modified version:

$$P(A) = \lim_{n \to \infty} \frac{\text{number of occurrences of } A \text{ amongst } n \text{ possibilities}}{n}. \tag{6.31}$$

Using this formula, in Problem 7, you should be able to prove that the probability a randomly chosen natural number is divisible by k is $1/k$. However, instead of using (6.31), we shall employ the following heuristic trick (which works to give the correct answer). Choose an "extremely large" natural number N, and consider the very large sample of numbers

$$1, 2, 3, 4, 5, 6, \ldots, Nk.$$

There are exactly N numbers in this list that are divisible by k, namely the N numbers $k, 2k, 3k, \ldots, Nk$, and no others, and there are a total of Nk numbers in this list. Thus, the probability that a natural number n, randomly chosen amongst the large sample, is divisible by k is exactly the probability that n is one of the N numbers $k, 2k, 3k, \ldots, Nk$, so

$$P(k|n) = \frac{\text{number of occurrences of divisibility}}{\text{total number of possibilities listed}} = \frac{N}{Nk} = \frac{1}{k}. \tag{6.32}$$

For instance, the probability that a randomly chosen natural number is divisible by 1 is 1, which makes sense. The probability that a randomly chosen natural number is divisible by 2 is 1/2; in other words, the probability that a randomly chosen natural number is even is 1/2, which also makes sense.

We are now ready to solve our two problems. Question: What is the probability that a natural number, chosen at random, is square free? Let $n \in \mathbb{N}$ be randomly chosen. Then n is square free just means that $p^2 \nmid n$ for all primes p. Thus,

$$P(n \text{ is square free}) = P((2^2 \nmid n) \text{ and } (3^2 \nmid n) \text{ and } (5^2 \nmid n) \text{ and } (7^2 \nmid n) \text{ and } \cdots).$$

Since n was randomly chosen, the events $2^2 \nmid n$, $3^2 \nmid n$, $5^2 \nmid n$, etc. are unrelated, so by (6.30),

$$P(n \text{ is square free}) = P(2^2 \nmid n) \cdot P(3^2 \nmid n) \cdot P(5^2 \nmid n) \cdot P(7^2 \nmid n) \cdots$$

To see what the right-hand side is, we use (6.29) and (6.32) to write

$$P(p^2 \nmid n) = 1 - P(p^2 | n) = 1 - \frac{1}{p^2}.$$

Thus,

$$P(n \text{ is square free}) = \prod_{p \text{ prime}} P(p^2 \nmid n) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2},$$

and our first question is answered!

Question: What is the probability that two given numbers, chosen at random, are relatively, or co, prime? Let $m, n \in \mathbb{N}$ be randomly chosen. Then m and n are **relatively prime**, or **coprime**, just means that m and n have no common factors (except 1), which means that $p \nmid$ both m, n for all prime numbers p.[2] Thus,

$$\begin{aligned} &P(m, n \text{ are relatively prime}) \\ &\qquad = P((2 \nmid \text{both } m, n) \text{ and } (3 \nmid \text{both } m, n) \text{ and } (5 \nmid \text{both } m, n) \text{ and } \cdots). \end{aligned}$$

Since m and n were randomly chosen, that $p \nmid$ both m, n is unrelated to $q \nmid$ both m, n, so by (6.30),

$$P(m, n \text{ are relatively prime}) = \prod_{p \text{ prime}} P(p \nmid \text{both } m, n).$$

[2]Explicitly, $p \nmid$ both m, n means $p \nmid m$ or $p \nmid n$.

To see what the right-hand side is, we use (6.29), (6.30), and (6.32) to write

$$P(p \not| \text{ both } m,n) = 1 - P(p| \text{ both } m,n) = 1 - P(p|m \text{ and } p|n)$$
$$= 1 - P(p|m)\cdot P(p|n) = 1 - \frac{1}{p}\cdot\frac{1}{p} = 1 - \frac{1}{p^2}.$$

Thus,

$$P(m,n \text{ are relatively prime}) = \prod_{p \text{ prime}} P(p \not| \text{ both } m,n) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right) = \frac{6}{\pi^2},$$

and our second question is answered!

Exercises 6.7.

1. We prove Theorem 6.16 using the good ole Tannery's theorem for products.
   (i) Let $r > 1$ be arbitrary and let $\operatorname{Re} z \geq r$. Prove that
   $$\left|\prod_{p<N} \frac{p^z - (1/p^z)^N}{p^z - 1} - \sum_{n=1}^{\infty} \frac{1}{n^z}\right| \leq \left|\sum_{n=N+1}^{\infty} \frac{1}{n^z}\right| \leq \sum_{n=N+1}^{\infty} \frac{1}{n^r}.$$
   Suggestion: $\frac{p^z-(1/p^z)^N}{p^z-1} = \frac{1-(1/p^z)^{N+1}}{1-1/p^z} = 1 + 1/p^z + 1/p^{2z} + \cdots + 1/p^{Nz}$.
   (ii) Write $\frac{p^z-(1/p^z)^N}{p^z-1} = 1 + \frac{1-(1/p^z)^N}{p^z-1}$. Show that
   $$\left|\frac{1-(1/p^z)^N}{p^z-1}\right| \leq \frac{2}{p^r - 1} \leq \frac{4}{p^r}$$
   and $\sum 4/p^r$ converges. Now prove Theorem 6.16 using Tannery's theorem for products.
2. Prove that for $z \in \mathbb{C}$ with $\operatorname{Re} z > 1$,
   $$\boxed{\frac{\zeta(z)}{\zeta(2z)} = \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^z}.}$$
   Suggestion: Show that $\frac{\zeta(z)}{\zeta(2z)} = \prod\left(1 + \frac{1}{p^z}\right)$ and copy the proof of Theorem 6.17.
3. (**Liouville's function**) Define
   $$\lambda(n) := \begin{cases} 1 & \text{if } n = 1 \\ 1 & \text{if the number of prime factors of } n \text{, counted with repetitions, is even} \\ -1 & \text{if the number of prime factors of } n \text{, counted with repetitions, is odd.} \end{cases}$$
   This function is called **Liouville's function** after Joseph Liouville (1809–1882). Prove that for $z \in \mathbb{C}$ with $\operatorname{Re} z > 1$,
   $$\boxed{\frac{\zeta(2z)}{\zeta(z)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^z}.}$$
   Suggestion: Show that $\frac{\zeta(2z)}{\zeta(z)} = \prod\left(1 + \frac{1}{p^z}\right)^{-1}$ and copy the proof of Theorem 6.16.
4. For $n \in \mathbb{N}$, let $\tau(n)$ denote the number of positive divisors of n (that is, the number of positive integers that divide n). Prove that for $z \in \mathbb{C}$ with $\operatorname{Re} z > 1$,
   $$\boxed{\zeta(z)^2 = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^z}.}$$
   Suggestion: By absolute convergence, we can write $\zeta(z)^2 = \sum_{m,n} 1/(m\cdot n)^z$ where this double series can be summed in any way we wish. Use Theorem 5.26 with the set S_k given by $S_k = T_1 \cup \cdots \cup T_k$ where $T_k = \{(m,n) \in \mathbb{N}\times\mathbb{N}\,;\, m\cdot n = k\}$.

5. Let $\zeta(z,a) := \sum_{n=0}^{\infty}(n+a)^{-z}$ for $z \in \mathbb{C}$ with $\operatorname{Re} z > 1$ and $a > 0$ — this function is called the **Hurwitz zeta function** after Adolf Hurwitz (1859–1919). Prove that

$$\sum_{m=1}^{k} \zeta\Big(z, \frac{m}{k}\Big) = k^z \zeta(z).$$

6. In this problem, we find useful bounds and limits for $\zeta(x)$ with $x > 1$ real.
   (a) Prove that $1 - \frac{1}{2^x} < \eta(x) < 1$.
   (b) Prove that
   $$\frac{1-2^{-x}}{1-2^{1-x}} < \zeta(x) < \frac{1}{1-2^{1-x}}.$$
   (c) Prove the following limits: $\zeta(x) \to 1$ as $x \to \infty$, $\zeta(x) \to \infty$ as $x \to 1^+$, and $(x-1)\zeta(x) \to 1$ as $x \to 1^+$.
7. Using the definition (6.31), prove that given a natural number k, the probability that a randomly chosen natural number is divisible by k is $1/k$ as follows. Amongst the n natural numbers $1, 2, 3, \ldots, n$, show that $\lfloor n/k \rfloor$ many numbers are divisible by k. Now take $n \to \infty$ in $\lfloor n/k \rfloor / n$.

6.8. ★ Some of the most beautiful formulæ in the world I

Hold on to your seats, for you're about to be taken on a journey through a beautiful world of mathematical formulas! In this section we rigorously derive a bunch of formulas that you'll find (derived in a somewhat nonrigorous fashion at least for today's standard) in Euler's wonderful book *Introduction to analysis of the infinite* [**52**] (the second book [**53**] is also great!).

6.8.1. Bernoulli numbers and evaluating sums/products. We start our onslaught of beautiful formulæ with a formula for the zeta function at all even natural numbers: $\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}}$. To find such a formula, we begin with the partial fraction expansion of the cotangent from Section 6.5:

$$\pi z \cot \pi z = 1 + 2z^2 \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2} = 1 - 2\sum_{n=1}^{\infty} \frac{z^2}{n^2 - z^2}.$$

Next, we apply Cauchy's double series theorem to this sum. Let $z \in \mathbb{C}$ be near 0 and observe that

$$\frac{z^2}{n^2 - z^2} = \frac{z^2/n^2}{1 - z^2/n^2} = \sum_{k=1}^{\infty} \Big(\frac{z^2}{n^2}\Big)^k.$$

Therefore,

$$\pi z \cot \pi z = 1 - 2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \Big(\frac{z^2}{n^2}\Big)^k.$$

Since

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \Big|\frac{z^2}{n^2}\Big|^k = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \Big(\frac{|z|^2}{n^2}\Big)^k = \frac{1}{2}\Big(1 - \pi \cot \pi |z|\Big) < \infty,$$

by Cauchy's double series theorem, we have

$$\pi z \cot \pi z = 1 - 2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \Big(\frac{z^2}{n^2}\Big)^k = 1 - 2\sum_{k=1}^{\infty}\Big(\sum_{n=1}^{\infty} \frac{1}{n^{2k}}\Big) z^{2k}. \tag{6.33}$$

On the other hand, we recall from Section 5.8 that $z\cot z = \sum_{k=0}^{\infty}(-1)^k \frac{2^{2k}\,B_{2k}}{(2k)!}\,z^{2k}$ (for $|z|$ small), where the B_{2k}'s are the Bernoulli numbers. Replacing z with πz, we get

$$\pi z\cot \pi z = 1 + \sum_{k=1}^{\infty}(-1)^k \frac{2^{2k}\,B_{2k}}{(2k)!}\,\pi^{2k}\,z^{2k}.$$

Comparing this equation with (6.33) and using the identity theorem, we see that

$$-2\sum_{n=1}^{\infty}\frac{1}{n^{2k}} = (-1)^k \frac{2^{2k}\,B_{2k}}{(2k)!}\,\pi^{2k}, \qquad k = 1, 2, 3, \ldots.$$

Rewriting this slightly, we obtain Euler's famous result: For $k = 1, 2, 3, \ldots$,

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^{2k}} = (-1)^{k-1}\frac{(2\pi)^{2k}\,B_{2k}}{2(2k)!} \quad ; \quad \text{that is,}\ \ \zeta(2k) = (-1)^{k-1}\frac{(2\pi)^{2k}\,B_{2k}}{2(2k)!}.} \tag{6.34}$$

In particular, using the known values of the Bernoulli numbers found in Section 5.8, setting $k = 1$ and $k = 2$, we get

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2} \quad \textbf{(Euler's sum, Proof VIII)} \quad \text{and} \quad \frac{\pi^4}{90} = \sum_{n=1}^{\infty}\frac{1}{n^4}.$$

Notice that (6.34) shows that $\zeta(2k)$ is a rational number times π^{2k}; in particular, since π is transcendental (see Section 10.5) it follows that $\zeta(n)$ is transcendental for n even. One may ask if there are similar expressions like (6.34) for sums of the reciprocals of the *odd* powers (e.g. $\zeta(3) = \sum_{n=1}^{\infty}\frac{1}{n^3}$). Unfortunately, there are no known formulas! Moreover, it is not even known if $\zeta(n)$ is transcendental for n odd and in fact, of all odd numbers only $\zeta(3)$ is known without a doubt to be irrational; this was proven by Roger Apéry (1916–1994) in 1979 (see [**22**], [**174**])!

Using (6.34), we can derive many other pretty formulas. First, in Theorem 6.18 we proved that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^z} = (1 - 2^{1-z})\zeta(z), \quad z > 1.$$

In particular, setting $z = 2k$, we find that for $k = 1, 2, 3, \ldots$,

$$\boxed{\eta(2k) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{2k}} = (-1)^{k-1}\big(1 - 2^{1-2k}\big)\frac{(2\pi)^{2k}\,B_{2k}}{2(2k)!};} \tag{6.35}$$

what formulas do you get when you set $k = 1, 2$? Second, recall from Theorem 6.16 that

$$\sum_{n=1}^{\infty}\frac{1}{n^z} = \prod\frac{p^z}{p^z - 1} = \frac{2^z}{2^z-1}\cdot\frac{3^z}{3^z-1}\cdot\frac{5^z}{5^z-1}\cdot\frac{7^z}{7^z-1}\cdots \tag{6.36}$$

where the product is over all primes. In particular, setting $z = 2$, we get

$$\boxed{\frac{\pi^2}{6} = \frac{2^2}{2^2-1}\cdot\frac{3^2}{3^2-1}\cdot\frac{5^2}{5^2-1}\cdot\frac{7^2}{7^2-1}\cdot\frac{11^2}{11^2-1}\cdots} \tag{6.37}$$

and setting $z = 4$, we get

$$\boxed{\frac{\pi^4}{90} = \frac{2^4}{2^4-1}\cdot\frac{3^4}{3^4-1}\cdot\frac{5^4}{5^4-1}\cdot\frac{7^4}{7^4-1}\cdot\frac{11^4}{11^4-1}\cdots.}$$

Dividing these two formulas and using that

$$\frac{\frac{n^4}{n^4-1}}{\frac{n^2}{n^2-1}} = n^2 \cdot \frac{n^2-1}{n^4-1} = n^2 \cdot \frac{n^2-1}{(n^2-1)(n^2+1)} = \frac{n^2}{n^2+1},$$

we obtain

$$\boxed{\frac{\pi^2}{15} = \frac{2^2}{2^2+1} \cdot \frac{3^2}{3^2+1} \cdot \frac{5^2}{5^2+1} \cdot \frac{7^2}{7^2+1} \cdot \frac{11^2}{11^2+1} \cdots .} \tag{6.38}$$

Third, recall from Theorem 6.17 that

$$\frac{1}{\zeta(z)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^z},$$

where $\mu(n)$ is the Möbius function. In particular, setting $z = 2$, we find that

$$\boxed{\frac{6}{\pi^2} = 1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} + \frac{1}{10^2} - \frac{1}{11^2} + \cdots ;}$$

what formula do you get when you set $z = 4$?

6.8.2. Euler numbers and evaluating sums. We now derive a formula for the *alternating* sum of the odd natural numbers to odd powers:

$$1 - \frac{1}{3^{2k+1}} + \frac{1}{5^{2k+1}} - \frac{1}{7^{2k+1}} + \frac{1}{9^{2k+1}} - + \cdots , \qquad k = 0, 1, 2, 3, \ldots .$$

First try: To this end, let $|z| < 1$ and recall from Section 6.5 that

$$\frac{\pi}{4\cos\frac{\pi z}{2}} = \frac{1}{1^2 - z^2} - \frac{3}{3^2 - z^2} + \frac{5}{5^2 - z^2} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)}{(2n+1)^2 - z^2}. \tag{6.39}$$

Observe that

$$\frac{(2n+1)}{(2n+1)^2 - z^2} = \frac{1}{(2n+1)} \cdot \frac{1}{1 - \frac{z^2}{(2n+1)^2}} = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2n+1)^{2k+1}}. \tag{6.40}$$

Thus,

$$\frac{\pi}{4\cos\frac{\pi z}{2}} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} (-1)^n \frac{z^{2k}}{(2n+1)^{2k+1}}. \tag{6.41}$$

Just as we did in proving (6.33), we shall try to use Cauchy's double series theorem on this sum ... however, observe that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \left| (-1)^n \frac{z^{2k}}{(2n+1)^{2k+1}} \right| = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{|z|^{2k}}{(2n+1)^{2k+1}} = \sum_{n=0}^{\infty} \frac{(2n+1)}{(2n+1)^2 - |z|^2},$$

which diverges (because this series behaves like $\sum \frac{1}{2n+1} = \infty$)! Therefore, we cannot apply Cauchy's double series theorem.

Second try: Let us start fresh from scratch. This time, we break up (6.39) into sums over n even and n odd (just consider the sums with n replaced by $2n$ and also by $2n + 1$):

$$\frac{\pi}{4\cos\frac{\pi z}{2}} = \sum_{n=0}^{\infty} \left(\frac{(4n+1)}{(4n+1)^2 - z^2} - \frac{(4n+3)}{(4n+3)^2 - z^2} \right).$$

Let $|z| < 1$. Then writing $\frac{(4n+1)}{(4n+1)^2-z^2}$ and $\frac{(4n+3)}{(4n+3)^2-z^2}$ as geometric series (just as we did in (6.40)) we see that

$$\frac{\pi}{4\cos\frac{\pi z}{2}} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{z^{2k}}{(4n+1)^{2k+1}} - \frac{z^{2k}}{(4n+3)^{2k+1}}\right)$$

$$(6.42) \qquad = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{1}{(4n+1)^{2k+1}} - \frac{1}{(4n+3)^{2k+1}}\right)z^{2k}.$$

We can now use Cauchy's double series theorem on this sum because

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left|\left(\frac{1}{(4n+1)^{2k+1}} - \frac{1}{(4n+3)^{2k+1}}\right)z^{2k}\right|$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{1}{(4n+1)^{2k+1}} - \frac{1}{(4n+3)^{2k+1}}\right)|z|^{2k} = \frac{\pi}{4\cos\frac{\pi|z|}{2}} < \infty,$$

where we used (6.42) with z replaced by $|z|$. Thus, by Cauchy's double series theorem, we have

$$\frac{\pi}{4\cos\frac{\pi z}{2}} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{1}{(4n+1)^{2k+1}} - \frac{1}{(4n+3)^{2k+1}}\right)z^{2k}.$$

We can combine the middle terms as

$$(6.43) \qquad \frac{\pi}{4\cos\frac{\pi z}{2}} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{(2n+1)^{2k+1}}\right)z^{2k};$$

thus, we can interchange orders in (6.41), but to justify it with complete mathematical rigor, we needed a little bit of mathematical gymnastics.

Now recall from Section 5.8 that

$$\frac{1}{\cos z} = \sec z = \sum_{k=0}^{\infty}(-1)^k\frac{E_{2k}}{(2k)!}\,z^{2k},$$

where the E_{2k}'s are the Euler numbers. Replacing z with $\pi z/2$ and multiplying by $\pi/4$, we get

$$\frac{\pi}{4\cos\frac{\pi z}{2}} = \frac{\pi}{4}\sum_{k=0}^{\infty}(-1)^k\frac{E_{2k}}{(2k)!}\left(\frac{\pi}{2}\right)^{2k} z^{2k}.$$

Comparing this equation with (6.43) and using the identity theorem, we conclude that for $k = 0, 1, 2, 3, \ldots$,

$$(6.44) \qquad \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2k+1}} = (-1)^k\frac{E_{2k}}{2(2k)!}\left(\frac{\pi}{2}\right)^{2k+1};}$$

what pretty formulas do you get when you set $k = 0, 1, 2$? (Here, you need the Euler numbers calculated in Section 5.8.) We can derive many other pretty formulas from (6.44). To start this onslaught, we first state an "odd version" of Theorem 6.16:

Theorem 6.19. *For any $z \in \mathbb{C}$ with $\operatorname{Re} z > 1$, we have*

$$\boxed{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^z} = \frac{3^z}{3^z+1}\cdot\frac{5^z}{5^z-1}\cdot\frac{7^z}{7^z+1}\cdot\frac{11^z}{11^z+1}\cdot\frac{13^z}{13^z-1}\cdots,}$$

where the product is over odd primes (all primes except 2*) and where the* $\pm$ *signs in the denominators depends on whether the prime is of the form* $4k+3$ *(+ sign) or* $4k+1$ *(− sign), where* $k = 0, 1, 2, \ldots$*.*

Since the proof of this theorem is similar to that of Theorem 6.16, we shall leave the proof of this theorem to the interested reader; see Problem 5. In particular, setting $z = 1$, we get

$$\boxed{\frac{\pi}{4} = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \frac{23}{24} \cdots .} \tag{6.45}$$

The numerators of the fractions on the right are the odd prime numbers and the denominators are even numbers divisible by four and differing from the numerators by one. In (6.37), we found that

$$\frac{\pi^2}{6} = \frac{2^2}{2^2-1} \cdot \frac{3^2}{3^2-1} \cdot \frac{5^2}{5^2-1} \cdots = \frac{4}{3} \cdot \frac{3 \cdot 3}{2 \cdot 4} \cdot \frac{5 \cdot 5}{4 \cdot 6} \cdot \frac{7 \cdot 7}{6 \cdot 8} \cdot \frac{11 \cdot 11}{10 \cdot 12} \cdot \frac{13 \cdot 13}{12 \cdot 14} \cdots$$

Dividing this expression by (6.45), and cancelling like terms, we obtain

$$\frac{4\pi}{6} = \frac{\pi^2/6}{\pi/4} = \frac{4}{3} \cdot \frac{3}{2} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdots .$$

Multiplying both sides by 3/4, we get another one of Euler's famous formulas:

$$\boxed{\frac{\pi}{2} = \frac{3}{2} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \frac{23}{22} \cdots .} \tag{6.46}$$

The numerators of the fractions are the odd prime numbers and the denominators are even numbers not divisible by four and differing from the numerators by one. (6.45) and (6.46) are two of my favorite infinite product expansions for π.

6.8.3. Benoit Cloitre's e and π in a mirror. In this section we prove a unbelievable fact connecting e and π that is due to Benoit Cloitre [**143**], [**150**]. Define sequences $\{a_n\}$ and $\{b_n\}$ by $a_1 = b_1 = 0$, $a_2 = b_2 = 1$, and the rest as the following "mirror images":

$$a_{n+2} = a_{n+1} + \frac{1}{n} a_n$$
$$b_{n+2} = \frac{1}{n} b_{n+1} + b_n.$$

We shall prove that

$$\boxed{e = \lim_{n\to\infty} \frac{n}{a_n} \qquad , \qquad \frac{\pi}{2} = \lim_{n\to\infty} \frac{n}{b_n^2}.} \tag{6.47}$$

The sequences $\{a_n\}$ and $\{b_n\}$ look so similar and so do $\{\frac{n}{a_n}\}$ and $\{\frac{n}{b_n^2}\}$, yet they generate very different numbers. Seeing such a connection between e and π, which *a priori* are very different, makes you wonder if there isn't someone behind this "coincidence."

To prove the formula for e, let us define a sequence $\{s_n\}$ by $s_n = a_n/n$. Then $s_1 = a_1/1 = 0$ and $s_2 = a_2/2 = 1/2$. Observe that for $n \geq 2$, we have

$$\begin{aligned} s_{n+1} - s_n = \frac{a_{n+1}}{n+1} - \frac{a_n}{n} &= \frac{1}{n+1}\Big(a_{n+1} - \frac{n+1}{n}a_n\Big) \\ &= \frac{1}{n+1}\Big(a_n + \frac{1}{n-1}a_{n-1} - \Big(1 + \frac{1}{n}\Big)a_n\Big) \\ &= \frac{1}{n+1}\Big(\frac{1}{n-1}a_{n-1} - \frac{a_n}{n}\Big) \\ &= \frac{-1}{n+1}(s_n - s_{n-1}). \end{aligned}$$

Using induction we see that

$$\begin{aligned} s_{n+1} - s_n = \frac{-1}{n+1}(s_n - s_{n-1}) &= \frac{-1}{n+1} \cdot \frac{-1}{n}(s_{n-1} - s_{n-2}) \\ &= \frac{-1}{n+1} \cdot \frac{-1}{n} \cdot \frac{-1}{n-1}(s_{n-2} - s_{n-3}) = \cdots \text{ etc.} \\ &= \frac{-1}{n+1} \cdot \frac{-1}{n} \cdot \frac{-1}{n-1} \cdots \frac{-1}{3}(s_2 - s_1) \\ &= \frac{-1}{n+1} \cdot \frac{-1}{n} \cdot \frac{-1}{n-1} \cdots \frac{-1}{3} \cdot \frac{1}{2} = \frac{(-1)^{n-3}}{(n+1)!} = \frac{(-1)^{n+1}}{(n+1)!}. \end{aligned}$$

Thus, writing as a telescoping sum, we obtain

$$s_n = s_1 + \sum_{k=2}^{n}(s_k - s_{k-1}) = 0 + \sum_{k=2}^{n} \frac{(-1)^k}{k!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!},$$

which is exactly the n-th partial sum for the series expansion of e^{-1}. It follows that $s_n \to e^{-1}$ and so,

$$\lim_{n\to\infty} \frac{n}{a_n} = \lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{e^{-1}} = e,$$

as we claimed. The limit for π in (6.47) will be left to you (see Problem 2).

Exercises 6.8.

1. In this problem we derive other neat formulas:
   (1) Dividing (6.38) by $\pi^2/6$, prove that

$$\boxed{\frac{5}{2} = \frac{2^2+1}{2^2-1} \cdot \frac{3^2+1}{3^2-1} \cdot \frac{5^2+1}{5^2-1} \cdot \frac{7^2+1}{7^2-1} \cdot \frac{11^2+1}{11^2-1} \cdots,}$$

   quite a neat expression for 2.5.
   (2) Dividing (6.46) by (6.45), prove that

$$\boxed{2 = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{9} \cdot \frac{10}{9} \cdot \frac{12}{11} \cdots,}$$

   quite a neat expression for 2. The fractions on the right are formed as follows: Given an odd prime $3, 5, 7, \ldots$, we take the pair of even numbers immediately above and below the prime, divide them by two, then put the resulting even number as the numerator and the odd number as the denominator.
2. In this problem, we prove the limit for π in (6.47).

(i) Define $t_n = b_{n+1}/b_n$ for $n = 2, 3, 4, \ldots$. Prove that (for $n = 2, 3, 4, \ldots$), $t_{n+1} = 1/n + 1/t_n$ and then,

$$t_n = \begin{cases} 1 & n \text{ even} \\ \frac{n}{n-1} & n \text{ odd.} \end{cases}$$

(ii) Prove that $b_n^2 = t_2^2 \cdot t_3^2 \cdot t_4^2 \cdots t_{n-1}^2$, then using Wallis' formula, derive the limit for π in (6.47).

3. From Problem 6 in Exercises 6.7, prove that

$$\frac{2(2n)!\,(1-2^{2n})}{(2\pi)^{2n}\,(1-2^{1-2n})} < \big|B_{2n}\big| < \frac{2(2n)!}{(2\pi)^{2n}\,(1-2^{1-2n})}.$$

This estimate shows that the Bernoulli numbers grow incredibly fast as $n \to \infty$.

4. (**Radius of convergence**) In this problem we (finally) determine the radii of convergence of

$$z \cot z = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}\,B_{2n}}{(2n)!}\, z^{2n} \quad , \quad \tan z = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n}(2^{2n}-1)\,B_{2n}}{(2n)!}\, z^{2n-1} \ .$$

(a) Let $a_{2n} = (-1)^n \frac{2^{2n}\,B_{2n}}{(2n)!}$. Prove that

$$\lim_{n\to\infty} |a_{2n}|^{1/2n} = \lim_{n\to\infty} \frac{1}{\pi} \cdot 2^{1/2n} \cdot \zeta(2n)^{1/2n} = \frac{1}{\pi}.$$

Conclude that the radius of convergence of $z \cot z$ is π.

(b) Using a similar argument, show that the radius of convergence of $\tan z$ is $\pi/2$.

5. In this problem, we prove Theorem 6.19

(i) Let us call an odd number "type I" if it is of the form $4k+1$ for some $k = 0, 1, \ldots$ and "type II" if it is of the form $4k+3$ for some $k = 0, 1, \ldots$. Prove that every odd number is either of type I or type II.

(ii) Prove that type I $\times$ type I $=$ type I, type I $\times$ type II $=$ type II, and type II $\times$ type II $=$ type I.

(iii) Let $a, b, \ldots, c \in \mathbb{N}$ be odd. Prove that if there is an *odd* number of type II integers amongst $a, b, \ldots, c$, then $a \cdot b \cdots c$ is of type II, otherwise, $a \cdot b \cdots c$ is type I.

(iv) Show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^z} = \sum_{n=0}^{\infty} \frac{1}{(4n+1)^z} - \sum_{n=0}^{\infty} \frac{1}{(4n+3)^z},$$

a sum of type I and type II natural numbers!

(v) Let $z \in \mathbb{C}$ with $\operatorname{Re} z \geq r > 1$, let $1 < N \in \mathbb{N}$, and let $3 < 5 < \cdots < m < 2N+1$ be the odd prime numbers less than $2N+1$. In a similar manner as in the proof of Theorem 6.16, show that

$$\Big|\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^z} - \frac{3^z}{3^z+1} \cdot \frac{5^z}{5^z-1} \cdot \frac{7^z}{7^z+1} \cdots \frac{m^z}{m^z \pm 1} \cdot \frac{13^z}{13^z-1}\Big| \leq \sum_{n=N}^{\infty} \Big|\frac{1}{(2n+1)^z}\Big| \leq \sum_{n=N}^{\infty} \frac{1}{(2n+1)^r},$$

where the $+$ signs in the product are for type II odd primes and the $-$ signs for type I odd primes. Now finish the proof of Theorem 6.19.