CHAPTER 7

Infinite continued fractions

From time immemorial, the infinite has stirred men's emotions more than any other question. Hardly any other idea has stimulated the mind so fruitfully ... In a certain sense, mathematical analysis is a symphony of the infinite.
David Hilbert (1862-1943) "On the infinite" [**20**].

We dabbed a little into the theory of continued fractions (that is, fractions that continue on and on and on ...) way back in the exercises of Section 3.4. In this chapter we concentrate on this fascinating subject. We start in Section 7.1 by showing that such fractions occur very naturally in long division and we give their basic definitions. In Section 7.2, we prove some pretty dramatic formulas (this is one reason continued fractions are so fascinating, at least to me). For example, we'll show that $4/\pi$ and π can be written as the continued fractions:

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \ddots}}}} \quad , \qquad \pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \cfrac{7^2}{6 + \ddots}}}}.$$

The continued fraction on the left is due to Lord Brouncker (and is the first continued fraction ever recorded) and the one on the right is due to Euler. If you think these π formulas are cool, we'll derive the following formulas for e as well:

$$e = 2 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \ddots}}}} = 1 + \cfrac{1}{0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \ddots}}}}}}}.$$

We'll prove the formula on the left in Section 7.2, but you'll have to wait for the formula on the right until Section 7.7. In Section 7.3, we discuss elementary properties of continued fractions. In this section we also discuss how a Greek mathematician, Diophantus of Alexandrea ($\approx$ 200–284 A.D.), can help you if you're stranded on an island with guys you can't trust and a monkey with a healthy appetite! In Section 7.4 we study the convergence properties of continued fractions.

Recall from our discussion on the amazing number π and its computations from ancient times (see Section 4.10) that throughout the years, the following approximation to π came up: 3, 22/7, 333/106, and 355/113. Did you ever wonder why these particular numbers occur? Also, did you ever wonder why our calendar is constructed the way it is (e.g. why leap years occur)? Finally, did you ever wonder why a piano has twelve keys (within an octave)? In Sections 7.5 and 7.6 you'll find out that these mysteries have to do with continued fractions! In Section 7.8 we study special types of continued fractions having to do with square roots and in Section 7.9 we learn why Archimedes needed around 8×10^{206544} cattle in order to "have abundant of knowledge in this science [mathematics]"!

In the very last section of Book I, Section 7.10, we look at continued fractions and transcendental numbers. We also ready ourselves for Book II, where we shall learn the celebrated calculus. (Isn't it amazing how much we've accomplished in Book I, all without taking a single derivative or integral!)

Chapter 7 objectives: The student will be able to ...

- define a continued fraction, state the Wallis-Euler and fundamental recurrence relations, and apply the continued fraction convergence theorem (Theorem 7.14).
- compute the canonical continued fraction of a given real number.
- understand the relationship between convergents of a simple continued fraction and best approximations, and the relationship between periodic simple continued fractions and quadratic irrationals.
- solve simple diophantine equations (of linear and Pell type).

7.1. Introduction to continued fractions

In this section we introduce the basics of continued fractions and see how they arise out of high school division and also from solving equations.

7.1.1. Continued fractions arise when dividing. A common way continued fractions arise is through "repeated divisions".

Example 7.1. Take for instance, high school division of 68 into 157: $\frac{157}{68} = 2 + \frac{21}{68}$. Inverting the fraction $\frac{21}{68}$, we can write $\frac{157}{68}$ as

$$\frac{157}{68} = 2 + \cfrac{1}{\cfrac{68}{21}}.$$

Since we can further divide $\frac{68}{21} = 3 + \frac{5}{21} = 3 + \frac{1}{21/5}$, we can write $\frac{157}{68}$ in the somewhat fancy way

$$\frac{157}{68} = 2 + \cfrac{1}{3 + \cfrac{1}{\cfrac{21}{5}}}.$$

Since $\frac{21}{5} = 4 + \frac{1}{5}$, we can write

$$\frac{157}{68} = 2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{5}}}. \tag{7.1}$$

Since 5 is now a whole number, our repeated division process stops.

The expression on the right in (7.1) is called a **finite simple continued fraction**. There are many ways to denote the right-hand side, but we shall stick with the following two:

$$\langle 2; 3, 4, 5\rangle \quad \text{or} \quad 2 + \frac{1}{3+}\,\frac{1}{4+}\,\frac{1}{5} \qquad \text{represent} \quad 2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{5}}}.$$

Thus, continued fractions (that is, fractions that "continue on") arise naturally out of writing rational numbers in a somewhat fancy way by repeated divisions. Of course, 157 and 68 were not special, by repeated divisions one can take *any* two integers a and b with $a \neq 0$ and write b/a as a finite simple continued fraction; see Problem 2. In Section 7.4, we shall prove that any real number, not necessarily rational, can be expressed as a simple (possibly infinite) continued fraction.

7.1.2. Continued fractions arise when solving equations. Continued fractions also arise naturally when trying to solve equations.

Example 7.2. Suppose we want to find the positive solution x to the equation $x^2 - x - 2 = 0$. Notice that 2 is the only positive solution. On the other hand, writing $x^2 - x - 2 = 0$ as $x^2 = x + 2$ and dividing by x, we get

$$x = 1 + \frac{2}{x} \quad \text{or, since } x = 2, \quad 2 = 1 + \frac{2}{x}.$$

We can replace x in the denominator with $x = 1 + 2/x$ to get

$$2 = 1 + \cfrac{2}{1 + \cfrac{2}{x}}.$$

Repeating this many times, we can write

$$2 = 1 + \cfrac{2}{1 + \cfrac{2}{1 + \cfrac{2}{\ddots\; 1 + \cfrac{2}{1 + \cfrac{2}{x}}}}}.$$

Repeating this "to infinity", we write

$$\text{“}\quad 2 = 1 + \cfrac{2}{1 + \cfrac{2}{1 + \cfrac{2}{1 + \cfrac{2}{1 + \ddots}}}}. \quad \text{”}$$

Quite a remarkable formula for 2! Later, (see Problem 4 in Section 7.4) we shall see that *any* whole number can be written in such a way. The reason for the quotation marks is that we have not yet defined what the right-hand object means.

We shall define what this means in a moment, but before doing so, here's another neat example:

Example 7.3. Consider the slightly modified formula $x^2 - x - 1 = 0$. Then $\Phi = \frac{1+\sqrt{5}}{2}$, called the **golden ratio**, is the only positive solution. We can rewrite $\Phi^2 - \Phi - 1 = 0$ as $\Phi = 1 + \frac{1}{\Phi}$. Replacing Φ in the denominator with $\Phi = 1 + \frac{1}{\Phi}$, we get

$$\Phi = 1 + \cfrac{1}{1 + \cfrac{1}{\Phi}}.$$

Repeating this substitution process "to infinity", we can write

$$\text{“}\quad \Phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}\quad ,\text{”} \tag{7.2}$$

quite a beautiful expression! As a side remark, there are many false rumors concerning the golden ratio; see [**114**] for the rundown.

7.1.3. Basic definitions for continued fractions. In general, a fraction written as

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{\ddots}{a_{n-1} + \cfrac{b_n}{a_n}}}}} \tag{7.3}$$

where the a_k's and b_k's are real numbers. (Of course, we are implicitly assuming that these fractions are all well-defined, e.g. no divisions by zero are allowed. Also, when you simplify this big fraction by combining fractions, you need to go from the bottom up.) Notice that if $b_m = 0$ for some m, then

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{\ddots}{a_{n-1} + \cfrac{b_n}{a_n}}}}} = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{\ddots}{a_{m-2} + \cfrac{b_{m-1}}{a_{m-1}}}}}, \tag{7.4}$$

since the $b_m = 0$ will zero out everything below it. The continued fraction is called **simple** if all the b_k's are 1 and the a_k's are integers with a_k positive for $k \geq 1$. Instead of writing the continued fraction as we did above, which takes up a lot of space, we shall shorten it to:

$$a_0 + \frac{b_1}{a_1 +}\,\frac{b_2}{a_2 +}\,\frac{b_3}{a_3 +}\,\cdots\,\frac{b_n}{+\,a_n}.$$

In the simple fraction case, we shorten the notation to

$$a_0 + \frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\,\cdots\,\frac{1}{+\,a_n} = \langle a_0; a_1, a_2, a_3, \ldots, a_n\rangle.$$

If $a_0 = 0$, some authors write $\langle a_1, a_2, \ldots, a_n\rangle$ instead of $\langle 0; a_1, \ldots a_n\rangle$.

We now discuss infinite continued fractions. Let $\{a_n\}$, $n = 0, 1, 2, \ldots$, and $\{b_n\}$, $n = 1, 2, \ldots$, be sequences of real numbers and suppose that

$$c_n := a_0 + \frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\,\frac{b_3}{a_3+}\,\cdots\,\frac{b_n}{+\,a_n}$$

is defined for all n. We call c_n the **n-th convergent** of the continued fraction. If the limit, $\lim c_n$, exists, then we say that the **infinite continued fraction**

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \ddots}}} \quad \text{or} \quad a_0 + \frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\,\frac{b_3}{a_3+}\,\cdots \tag{7.5}$$

converges and we use either of these notations to denote the limiting value $\lim c_n$. In the simple continued fraction case (all the b_n's are 1 and the a_n's are integers with a_n natural for $n \geq 1$), in place of (7.5) we use the notation

$$\langle a_0; a_1, a_2, a_3, \ldots\rangle := \lim_{n\to\infty} \langle a_0; a_1, a_2, a_3, \ldots, a_n\rangle,$$

provided that the right-hand side exists. In particular, in Section 7.4 we shall prove that (7.2) does hold true:

$$2 = 1 + \frac{2}{1+}\,\frac{2}{1+}\,\frac{2}{1+}\,\cdots.$$

In the case when there is some b_m term that vanishes, then convergence of (7.5) is easy because (see (7.4)) for $n \geq m$, we have $c_n = c_{m-1}$. Hence, in this case

$$a_0 + \frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\,\frac{b_3}{a_3+}\,\cdots = a_0 + \frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\,\frac{b_3}{a_3+}\,\cdots\,\frac{b_{m-1}}{+\,a_{m-1}}$$

converges automatically (such a continued fraction is said to **terminate** or be **finite**). However, general convergence issues are not so straightforward. We shall deal with the subtleties of convergence in Section 7.4.

EXERCISES 7.1.

1. Expand the following fractions into finite simple continued fractions:
$$(a)\ \frac{7}{11}\quad,\quad (b)\ -\frac{11}{7}\quad,\quad (c)\ \frac{3}{13}\quad,\quad (d)\ \frac{13}{3}\quad,\quad (e)\ -\frac{42}{31}.$$
2. Prove that a real number can be written as a finite simple continued fraction if and only if it is rational. Suggestion: for the "if" statement, use the division algorithm (see Theorem 2.15): For $a, b \in \mathbb{Z}$ with $a > 0$, we have $b = qa + r$ where $q, r \in \mathbb{Z}$ with $0 \leq r < a$; if a, b are both nonnegative, then so is q.
3. Let $\xi = a_0 + \frac{b_1}{a_1+}\frac{b_2}{a_2+}\frac{b_3}{a_3+}\cdots\frac{b_n}{+\,a_n} \neq 0$. Prove that
$$\frac{1}{\xi} = \frac{1}{a_0+}\,\frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\,\frac{b_3}{a_3+}\,\cdots\,\frac{b_n}{+\,a_n}.$$
In particular, if $\xi = \langle a_0; a_1, \ldots, a_n\rangle \neq 0$, show that $\frac{1}{\xi} = \langle 0; a_0, a_1, a_2, \ldots, a_n\rangle$.

4. A useful technique to study continued fraction is the following artifice of writing a continued fraction within a continued fraction. For a continued fraction $\xi = a_0 + \frac{b_1}{a_1 +}\frac{b_2}{a_2 +}\frac{b_3}{a_3 +}\cdots\frac{}{+}\frac{b_n}{a_n}$, if $m < n$, prove that
$$\xi = a_0 + \frac{b_1}{a_1 +}\,\frac{b_2}{a_2 +}\,\frac{b_3}{a_3 +}\,\cdots\,\frac{b_m}{+\ \eta},\quad \text{where } \eta = \frac{b_{m+1}}{a_{m+1}}+\cdots+\frac{b_n}{a_n}.$$

7.2. ★ Some of the most beautiful formulæ in the world II

Hold on to your seats, for you're about to be taken on another journey through the beautiful world of mathematical formulas!

7.2.1. Transformation of continued fractions. It will often be convenient to transform one continued fraction to another one. For example, let ρ_1, ρ_2, ρ_3 be nonzero real numbers and suppose that the finite fraction
$$\xi = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3}}},$$
where the a_k's and b_k's are real numbers, is defined. Then multiplying the top and bottom of the fraction by ρ_1, we get
$$\xi = a_0 + \cfrac{\rho_1 b_1}{\rho_1 a_1 + \cfrac{\rho_1 b_2}{a_2 + \cfrac{b_3}{a_3}}}.$$
Multiplying the top and bottom of the fraction with $\rho_1 b_2$ as numerator by ρ_2 gives
$$\xi = a_0 + \cfrac{\rho_1 b_1}{\rho_1 a_1 + \cfrac{\rho_1 \rho_2 b_2}{\rho_2 a_2 + \cfrac{\rho_2 b_3}{a_3}}}.$$
Finally, multiplying the top and bottom of the fraction with $\rho_2 b_3$ as numerator by ρ_3 gives
$$\xi = a_0 + \cfrac{\rho_1 b_1}{\rho_1 a_1 + \cfrac{\rho_1 \rho_2 b_2}{\rho_2 a_2 + \cfrac{\rho_2 \rho_3 b_3}{\rho_3 a_3}}}.$$
In summary,
$$a_0 + \frac{b_1}{a_1 +}\,\frac{b_2}{a_2 +}\,\frac{b_3}{a_3} = a_0 + \frac{\rho_1 b_1}{\rho_1 a_1 +}\,\frac{\rho_1\rho_2 b_2}{\rho_2 a_2\ +}\,\frac{\rho_2\rho_3 b_3}{\rho_3 a_3}.$$
A simple induction argument proves the following.

Theorem 7.1 (**Transformation rules**). *For any real numbers $a_1, a_2, a_3, \ldots$, $b_1, b_2, b_3, \ldots$, and nonzero constants $\rho_1, \rho_2, \rho_3, \ldots$, we have*
$$a_0 + \frac{b_1}{a_1 +}\,\frac{b_2}{a_2 +}\,\frac{b_3}{a_3 +}\,\cdots\,\frac{b_n}{+\ a_n} = a_0 + \frac{\rho_1 b_1}{\rho_1 a_1 +}\,\frac{\rho_1\rho_2 b_2}{\rho_2 a_2\ +}\,\frac{\rho_2\rho_3 b_3}{\rho_3 a_3\ +}\,\cdots\,\frac{\rho_{n-1}\rho_n b_n}{+\ \rho_n a_n},$$

in the sense when the left-hand side is defined, so is the right-hand side and this equality holds. In particular, if the limit as $n \to \infty$ of the left-hand side exists, then the limit of the right-hand side also exists, and

$$a_0 + \frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\,\cdots\,\frac{b_n}{+\,a_n+}\,\cdots = a_0 + \frac{\rho_1 b_1}{\rho_1 a_1+}\,\frac{\rho_1\rho_2 b_2}{\rho_2 a_2\ +}\,\cdots\,\frac{\rho_{n-1}\rho_n b_n}{+\quad \rho_n a_n \quad +}\,\cdots.$$

7.2.2. Two stupendous series and continued fractions identities. Let $\alpha_1, \alpha_2, \alpha_3, \ldots$ be any real numbers with $\alpha_k \neq 0$ and $\alpha_k \neq \alpha_{k-1}$ for all k. Observe that

$$\frac{1}{\alpha_1} - \frac{1}{\alpha_2} = \frac{\alpha_2 - \alpha_1}{\alpha_1\alpha_2} = \frac{1}{\frac{\alpha_1\alpha_2}{\alpha_2-\alpha_1}}.$$

Since

$$\frac{\alpha_1\alpha_2}{\alpha_2 - \alpha_1} = \frac{\alpha_1(\alpha_2 - \alpha_1) + \alpha_1^2}{\alpha_2 - \alpha_1} = \alpha_1 + \frac{\alpha_1^2}{\alpha_2 - \alpha_1},$$

we get

$$\frac{1}{\alpha_1} - \frac{1}{\alpha_2} = \frac{1}{\alpha_1 + \frac{\alpha_1^2}{\alpha_2-\alpha_1}}.$$

This little exercise suggests the following theorem.

THEOREM 7.2. *If $\alpha_1, \alpha_2, \alpha_3, \ldots$ are nonzero real numbers with $\alpha_k \neq \alpha_{k-1}$ for all k, then for any $n \in \mathbb{N}$,*

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{\alpha_k} = \cfrac{1}{\alpha_1 + \cfrac{\alpha_1^2}{\alpha_2 - \alpha_1 + \cfrac{\alpha_2^2}{\ddots \atop \alpha_3 - \alpha_2 + \cfrac{\alpha_{n-1}^2}{\alpha_n - \alpha_{n-1}}}}}.$$

In particular, taking $n \to \infty$, we conclude that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\alpha_k} = \frac{1}{\alpha_1+}\,\frac{\alpha_1^2}{\alpha_2 - \alpha_1+}\,\frac{\alpha_2^2}{\alpha_3 - \alpha_2+}\,\frac{\alpha_3^2}{\alpha_4 - \alpha_3+}\cdots \tag{7.6}$$

as long as either (and hence both) side makes sense.

PROOF. This theorem is certainly true for alternating sums with $n = 1$ terms. Assume it is true for sums with n terms; we shall prove it holds for sums with $n+1$ terms. Observe that we can write

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{\alpha_k} &= \frac{1}{\alpha_1} - \frac{1}{\alpha_2} + \cdots + \frac{(-1)^{n-1}}{\alpha_n} + \frac{(-1)^n}{\alpha_{n+1}} \\
&= \frac{1}{\alpha_1} - \frac{1}{\alpha_2} + \cdots + (-1)^{n-1}\left(\frac{1}{\alpha_n} - \frac{1}{\alpha_{n+1}}\right) \\
&= \frac{1}{\alpha_1} - \frac{1}{\alpha_2} + \cdots + (-1)^{n-1}\left(\frac{\alpha_{n+1} - \alpha_n}{\alpha_n\alpha_{n+1}}\right) \\
&= \frac{1}{\alpha_1} - \frac{1}{\alpha_2} + \cdots + (-1)^{n-1}\frac{1}{\frac{\alpha_n\alpha_{n+1}}{\alpha_{n+1}-\alpha_n}}.
\end{aligned}$$

This is now a sum of n terms. Thus, we can apply the induction hypothesis to conclude that

$$\sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{\alpha_k} = \frac{1}{\alpha_1 +} \frac{\alpha_1^2}{\alpha_2 - \alpha_1 +} \cdots + \frac{\alpha_{n-1}^2}{\frac{\alpha_n \alpha_{n+1}}{\alpha_{n+1}-\alpha_n} - \alpha_{n-1}}. \tag{7.7}$$

Since

$$\begin{aligned}\frac{\alpha_n \alpha_{n+1}}{\alpha_{n+1} - \alpha_n} - \alpha_{n-1} &= \frac{\alpha_n(\alpha_{n+1} - \alpha_n) + \alpha_n^2}{\alpha_{n+1} - \alpha_n} - \alpha_{n-1} \\ &= \alpha_n - \alpha_{n-1} + \frac{\alpha_n^2}{\alpha_{n+1} - \alpha_n},\end{aligned}$$

putting this into (7.7) gives

$$\sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{\alpha_k} = \frac{1}{\alpha_1 +} \frac{\alpha_1^2}{\alpha_2 - \alpha_1 +} \cdots + \frac{\alpha_{n-1}^2}{\alpha_n - \alpha_{n-1} + \frac{\alpha_n^2}{\alpha_{n+1}-\alpha_n}}.$$

This proves our induction step and completes our proof. □

Example 7.4. Since we know that

$$\log 2 = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots,$$

setting $\alpha_k = k$ in (7.6), we can write

$$\boxed{\log 2 = \frac{1}{1+} \frac{1^2}{1+} \frac{2^2}{1+} \frac{3^2}{1+} \cdots,}$$

which we can also write as the equally beautiful expression

$$\boxed{\log 2 = \cfrac{1}{1 + \cfrac{1^2}{1 + \cfrac{2^2}{1 + \cfrac{3^2}{1 + \cfrac{4^2}{1 + \ddots}}}}}.}$$

See Problem 1 for a general formula for $\log(1 + x)$.

Here is another interesting identity. Let $\alpha_1, \alpha_2, \alpha_3, \ldots$ be real, nonzero, and never equal to 1. Then observe that

$$\frac{1}{\alpha_1} - \frac{1}{\alpha_1\alpha_2} = \frac{\alpha_2 - 1}{\alpha_1\alpha_2} = \frac{1}{\frac{\alpha_1\alpha_2}{\alpha_2 - 1}}.$$

Since

$$\frac{\alpha_1\alpha_2}{\alpha_2 - 1} = \frac{\alpha_1(\alpha_2 - 1) + \alpha_1}{\alpha_2 - 1} = \alpha_1 + \frac{\alpha_1}{\alpha_2 - 1},$$

we get

$$\frac{1}{\alpha_1} - \frac{1}{\alpha_1\alpha_2} = \frac{1}{\alpha_1 + \frac{\alpha_1}{\alpha_2 - 1}}.$$

We can continue by induction in much the same manner as we did in the proof of Theorem 7.2 to obtain the following result.

THEOREM 7.3. *For any real sequence $\alpha_1, \alpha_2, \alpha_3, \ldots$ with $\alpha_k \neq 0, 1$, we have*

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{\alpha_1 \cdots \alpha_k} = \cfrac{1}{\alpha_1 + \cfrac{\alpha_1}{\alpha_2 - 1 + \cfrac{\alpha_2}{\alpha_3 - 1 + \cfrac{\ddots}{\alpha_{n-1} + \cfrac{\alpha_{n-1}}{\alpha_n - 1}}}}}.$$

In particular, taking $n \to \infty$, we conclude that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\alpha_1 \cdots \alpha_k} = \frac{1}{\alpha_1 +} \frac{\alpha_1}{\alpha_2 - 1 +} \frac{\alpha_2}{\alpha_3 - 1 +} \cdots \frac{\alpha_{n-1}}{+ \alpha_n - 1} \cdots, \tag{7.8}$$

as long as either (and hence both) side makes sense.

Theorems 7.2 and 7.3 turn series to continued fractions; in Problem 9 we do the same for infinite products.

7.2.3. Continued fractions for arctan **and π.** We now use the identities just learned to derive some remarkable continued fractions.

Example 7.5. First, since

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots,$$

using the limit expression (7.6) in Theorem 7.2:

$$\frac{1}{\alpha_1} - \frac{1}{\alpha_2} + \frac{1}{\alpha_3} - \frac{1}{\alpha_4} + \cdots = \frac{1}{\alpha_1 +} \frac{\alpha_1^2}{\alpha_2 - \alpha_1 +} \frac{\alpha_2^2}{\alpha_3 - \alpha_2 +} \frac{\alpha_3^2}{\alpha_4 - \alpha_3 +} \cdots,$$

we can write

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \ddots}}}}}.$$

Inverting both sides (see Problem 3 in Exercises 7.1), we obtain the incredible expansion:

$$\boxed{\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \ddots}}}}.} \tag{7.9}$$

This continued fraction was the very first continued fraction ever recorded, and was written down without proof by Lord Brouncker (1620 – 1686), the first president of the Royal Society of London.

Actually, we can derive (7.9) from a related expansion of the arctangent function, which is so neat that we shall derive in two ways, using Theorem 7.2 then using Theorem 7.3.

Example 7.6. Recall that

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots$$

Setting $\alpha_1 = \frac{1}{x}$, $\alpha_2 = \frac{3}{x^3}$, $\alpha_3 = \frac{5}{x^5}$, and in general, $\alpha_n = \frac{2n-1}{x^{2n-1}}$ into the formula (7.6) from Theorem 7.2, we get the somewhat complicated formula

$$\arctan x = \frac{1}{\frac{1}{x}+}\ \frac{\frac{1}{x^2}}{\frac{3}{x^2} - \frac{1}{x}+}\ \frac{\frac{3^2}{x^2}}{\frac{5}{x^5} - \frac{3}{x^3}+}\ \cdots + \frac{\frac{(2n-3)^2}{(x^{2n-3})^2}}{\frac{2n-1}{x^{2n-1}} - \frac{2n-3}{x^{2n-3}}+}\ \cdots.$$

However, we can simplify this using the transformation rule from Theorem 7.1:

$$\frac{b_1}{a_1+}\ \frac{b_2}{a_2+}\ \cdots + \frac{b_n}{a_n+}\ \cdots = \frac{\rho_1 b_1}{\rho_1 a_1+}\ \frac{\rho_1\rho_2 b_2}{\rho_2 a_2\ +}\ \cdots + \frac{\rho_{n-1}\rho_n b_n}{\rho_n a_n\ +}\ \cdots.$$

(Here we drop the a_0 term from that theorem.) Let $\rho_1 = x$, $\rho_2 = x^3$, …, and in general, $\rho_n = x^{2n-1}$. Then,

$$\frac{1}{\frac{1}{x}+}\ \frac{\frac{1}{x^2}}{\frac{3}{x^2} - \frac{1}{x}+}\ \frac{\frac{3^2}{x^2}}{\frac{5}{x^5} - \frac{3}{x^3}+}\ \frac{\frac{5^2}{x^2}}{\frac{7}{x^7} - \frac{5}{x^5}+}\ \cdots = \frac{x}{1+}\ \frac{x^2}{3-x^2+}\ \frac{3^2x^2}{5-3x^2+}\ \frac{5^2x^2}{7-5x^2+}\ \cdots.$$

Thus,

$$\arctan x = \frac{x}{1+}\ \frac{x^2}{3-x^2+}\ \frac{3^2x^2}{5-3x^2+}\ \frac{5^2x^2}{7-5x^2+}\ \cdots,$$

or in a fancier way:

$$\text{(7.10)}\qquad \boxed{\arctan x = \cfrac{x}{1 + \cfrac{x^2}{(3-x^2) + \cfrac{3^2x^2}{(5-3x^2) + \cfrac{5^2x^2}{(7-5x^2) + \ddots}}}}.}$$

In particular, setting $x = 1$ and inverting, we get Lord Brouncker's formula:

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \ddots}}}}.$$

Example 7.7. We can also derive (7.10) using Theorem 7.3: Using once again that

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots$$

and setting $\alpha_1 = \frac{1}{x}$, $\alpha_2 = \frac{3}{x^2}$, $\alpha_3 = \frac{5}{3x^2}$, $\alpha_4 = \frac{7}{5x^2}$, $\cdots$, $\alpha_n = \frac{2n-1}{(2n-3)x^2}$ for $n \geq 2$, into the limiting formula (7.8) from Theorem 7.3:

$$\frac{1}{\alpha_1} - \frac{1}{\alpha_1\alpha_2} + \frac{1}{\alpha_1\alpha_2\alpha_3} - \cdots = \frac{1}{\alpha_1+}\ \frac{\alpha_1}{\alpha_2 - 1+}\ \frac{\alpha_2}{\alpha_3 - 1+}\ \cdots + \frac{\alpha_{n-1}}{\alpha_n - 1+}\ \cdots$$

we obtain

$$\arctan x = \cfrac{1}{\frac{1}{x} +} \; \cfrac{\frac{1}{x}}{\frac{3}{x^2} - 1 +} \; \cfrac{\frac{3}{x^2}}{\frac{5}{3x^2} - 1 +} \cdots + \cfrac{\frac{2n-1}{(2n-3)x^2}}{\frac{2n+1}{(2n-1)x^2} - 1 +} \cdots .$$

From Theorem 7.1, we know that

$$\frac{b_1}{a_1 +} \; \frac{b_2}{a_2 +} \cdots + \frac{b_n}{a_n +} \cdots = \frac{\rho_1 b_1}{\rho_1 a_1 +} \; \frac{\rho_1 \rho_2 b_2}{\rho_2 a_2 +} \cdots + \frac{\rho_{n-1} \rho_n b_n}{\rho_n a_n +} \cdots .$$

In particular, setting $\rho_1 = x$, $\rho_2 = x^2$, $\rho_3 = 3x^2$, $\rho_4 = 5x^2$, and in general, $\rho_n = (2n-3)x^2$ for $n \geq 1$ into the formula for $\arctan x$, we obtain (as you are invited to verify) the exact same expression (7.10)!

Example 7.8. We leave the next two beauts to you! Applying Theorem 7.2 and Theorem 7.3 to Euler's sum $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$, in Problem 2 we ask you to derive the formula

$$\boxed{\frac{6}{\pi^2} = 0^2 + 1^2 - \cfrac{1^4}{1^2 + 2^2 - \cfrac{2^4}{2^2 + 3^2 - \cfrac{3^4}{3^2 + 4^2 - \cfrac{4^4}{4^2 + 5^2 - \ddots}}}},} \tag{7.11}$$

which is, after inversion, the last formula on the front cover of this book.

Example 7.9. In Problem 9 we derive Euler's splendid formula [**37**, p. 89]:

$$\boxed{\frac{\pi}{2} = 1 + \cfrac{1}{1 + \cfrac{1 \cdot 2}{1 + \cfrac{2 \cdot 3}{1 + \cfrac{3 \cdot 4}{1 + \ddots}}}}.} \tag{7.12}$$

7.2.4. Another continued fraction for π. We now derive another remarkable formula for π, which is due to Euler [**37**, p. 89] (the proof we give is found in Lange's article [**99**]). Consider first the telescoping sum

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{1}{n} + \frac{1}{n+1} \right) = \left(\frac{1}{1} + \frac{1}{2} \right) - \left(\frac{1}{2} + \frac{1}{3} \right) + \left(\frac{1}{3} + \frac{1}{4} \right) - + \cdots = 1$$

Since

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = 1 - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1},$$

multiplying this expression by 4 and using the previous expression, we can write

$$\begin{aligned}\pi = 4 - 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n+1} &= 3 + 1 - 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n+1}\\ &= 3 + \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n} + \frac{1}{n+1}\right) - 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n+1}\\ &= 3 + \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n} + \frac{1}{n+1} - \frac{4}{2n+1}\right)\\ &= 3 + 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n(2n+1)(2n+2)},\end{aligned}$$

where we combined fractions in going from the third to forth lines. We now apply the limiting formula (7.6) from Theorem 7.2 with $\alpha_n = 2n(2n+1)(2n+2)$. Observe that

$$\begin{aligned}\alpha_n - \alpha_{n-1} &= 2n(2n+1)(2n+2) - 2(n-1)(2n-1)(2n)\\ &= 4n\big[(2n+1)(n+1) - (n-1)(2n-1)\big]\\ &= 4n\big[2n^2 + 2n + n + 1 - (2n^2 - n - 2n + 1)\big] = 4n(6n) = 24n^2.\end{aligned}$$

Now putting the α_n's into the formula

$$\frac{1}{\alpha_1} - \frac{1}{\alpha_2} + \frac{1}{\alpha_3} - \frac{1}{\alpha_4} + \cdots = \frac{1}{\alpha_1 +}\,\frac{\alpha_1^2}{\alpha_2 - \alpha_1 +}\,\frac{\alpha_2^2}{\alpha_3 - \alpha_2 +}\,\frac{\alpha_3^2}{\alpha_4 - \alpha_3 +}\cdots,$$

we get

$$\begin{aligned}4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n(2n+1)(2n+2)} &= 4\cdot\left(\frac{1}{2\cdot 3\cdot 4 +}\,\frac{(2\cdot 3\cdot 4)^2}{24\cdot 2^2 +}\,\frac{(4\cdot 5\cdot 6)^2}{24\cdot 3^2 +}\cdots\right)\\ &= \frac{1}{2\cdot 3 +}\,\frac{(2\cdot 3\cdot 4)^2}{24\cdot 2^2 +}\,\frac{(4\cdot 5\cdot 6)^2}{24\cdot 3^2 +}\cdots.\end{aligned}$$

Hence,

$$\pi = 3 + \frac{1}{6 +}\,\frac{(2\cdot 3\cdot 4)^2}{24\cdot 2^2 +}\,\frac{(4\cdot 5\cdot 6)^2}{24\cdot 3^2 +}\cdots\frac{(2(n-1)(2n-1)(2n))^2}{24\cdot n^2 +}\cdots,$$

which is beautiful, but we can make this even more beautiful using the transformation rule from Theorem 7.1:

$$\frac{b_1}{a_1 +}\,\frac{b_2}{a_2 +}\cdots\frac{b_n}{a_n +}\cdots = \frac{\rho_1 b_1}{\rho_1 a_1 +}\,\frac{\rho_1\rho_2 b_2}{\rho_2 a_2 +}\cdots\frac{\rho_{n-1}\rho_n b_n}{\rho_n a_n +}\cdots.$$

Setting $\rho_1 = 1$ and $\rho_n = \frac{1}{4n^2}$ for $n \geq 2$ we see that

$$\frac{\rho_{n-1}\rho_n b_n}{\rho_n a_n} = \frac{\frac{1}{4(n-1)^2}\cdot\frac{1}{4n^2}\cdot(2(n-1)(2n-1)(2n))^2}{\frac{1}{4n^2}\cdot 24\cdot n^2} = \frac{(2n-1)^2}{6}.$$

Thus,

$$\pi = 3 + \frac{1^2}{6 +}\,\frac{3^2}{6 +}\,\frac{5^2}{6 +}\,\frac{7^2}{6 +}\cdots\frac{(2n-1)^2}{6 +}\cdots$$

or in more elegant notation:

$$\boxed{\pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \cfrac{7^2}{6 + \ddots}}}}.} \tag{7.13}$$

7.2.5. Continued fractions and e. For our final beautiful example, we shall compute a continued fraction expansion for e. We begin with

$$\frac{1}{e} = e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - \frac{1}{1} + \frac{1}{1 \cdot 2} - \frac{1}{1 \cdot 2 \cdot 3} + \cdots ,$$

so

$$\frac{e-1}{e} = 1 - \frac{1}{e} = \frac{1}{1} - \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} - \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots .$$

Thus, setting $\alpha_k = k$ into the formula (7.8):

$$\frac{1}{\alpha_1} - \frac{1}{\alpha_1\alpha_2} + \frac{1}{\alpha_1\alpha_2\alpha_3} - \cdots = \frac{1}{\alpha_1 +} \frac{\alpha_1}{\alpha_2 - 1 +} \frac{\alpha_2}{\alpha_3 - 1 +} \cdots \frac{\alpha_{n-1}}{+ \alpha_n - 1 +} \cdots ,$$

we obtain

$$\frac{e-1}{e} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \ddots}}}} .$$

We can make in this into an expression for e as follows: First, invert and subtract 1 from both sides to get

$$\frac{e}{e-1} = 1 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \ddots}}} \quad \Longrightarrow \quad \frac{1}{e-1} = \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \ddots}}} .$$

Second, invert again to obtain

$$e - 1 = 1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \ddots}}}} .$$

Finally, adding 1 to both sides we get the incredibly beautiful expression

$$e = 2 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \ddots}}}}, \tag{7.14}$$

or in shorthand:

$$e = 2 + \frac{2}{2+}\,\frac{3}{3+}\,\frac{4}{4+}\,\frac{5}{5+}\cdots.$$

In the exercises you will derive other amazing formulæ.

Exercises 7.2.

1. Recall that $\log(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$. Using this formula, the formula (7.6) derived from Theorem 7.2, and also the transformation rule, prove that fabulous formula
$$\log(1+x) = \cfrac{x}{1 + \cfrac{1^2 x}{(2-1x) + \cfrac{2^2 x}{(3-2x) + \cfrac{3^2 x}{(4-3x) + \ddots}}}}.$$
Plug in $x = 1$ to derive our beautiful formula for $\log 2$.
2. Using Euler's sum $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$, give two proofs of the formula (7.11), one using Theorem 7.2 and the other using Theorem 7.3. The transformation rules will also come in handy.
3. (i) For any real numbers $\{\alpha_k\}$, prove that for any n,
$$\sum_{k=0}^{n} \alpha_k x^k = \alpha_0 + \frac{\alpha_1 x}{1} + \frac{-\frac{\alpha_2}{\alpha_1}x}{1 + \frac{\alpha_2}{\alpha_1}x +} \frac{-\frac{\alpha_3}{\alpha_2}x}{1 + \frac{\alpha_3}{\alpha_2}x +} \cdots + \frac{-\frac{\alpha_n}{\alpha_{n-1}}x}{1 + \frac{\alpha_n}{\alpha_{n-1}}x}$$
provided, of course, that the right-hand side is defined, which we assume holds for every n.
   (ii) Deduce that if the infinite series $\sum_{n=0}^{\infty} \alpha_n x^n$ converges, then
$$\sum_{n=0}^{\infty} \alpha_n x^n = \alpha_0 + \frac{\alpha_1 x}{1} + \frac{-\frac{\alpha_2}{\alpha_1}x}{1 + \frac{\alpha_2}{\alpha_1}x +} \frac{-\frac{\alpha_3}{\alpha_2}x}{1 + \frac{\alpha_3}{\alpha_2}x +} \cdots + \frac{-\frac{\alpha_n}{\alpha_{n-1}}x}{1 + \frac{\alpha_n}{\alpha_{n-1}}x +} \cdots.$$
Transforming the continued fraction on the right, prove that
$$\sum_{n=0}^{\infty} \alpha_n x^n = \alpha_0 + \frac{\alpha_1 x}{1} + \frac{-\alpha_2 x}{\alpha_1 + \alpha_2 x +} \frac{-\alpha_1\alpha_3 x}{\alpha_2 + \alpha_3 x +} \cdots + \frac{-\alpha_{n-2}\alpha_n x}{\alpha_{n-1} + \alpha_n x +} \cdots.$$
4. Writing $\arctan x = x(1 - \frac{y}{3} + \frac{y^2}{5} - \frac{y^3}{7} + \cdots)$ where $y = x^2$, and using the previous problem on $(1 - \frac{y}{3} + \frac{y^2}{5} - \frac{y^3}{7} + \cdots)$, derive the formula (7.10).
5. Let $x, y > 0$. Prove that
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{x+ny} = \frac{1}{x+}\,\frac{x^2}{y+}\,\frac{(x+y)^2}{y}+\frac{(x+2y)^2}{y}+\frac{(x+3y)^2}{y}+\cdots.$$
Suggestion: The formula (7.6) might help.
6. Recall the partial fraction expansion $\pi x \cot \pi x = 1 + 2x^2 \sum_{n=1}^{\infty} \frac{1}{x^2 - n^2}$.

(a) By breaking up $\frac{2x}{x^2-n^2}$ using partial fractions, prove that

$$\pi\cot\pi x = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - + \cdots .$$

(b) Derive the remarkable formula

$$\boxed{\pi\cot\pi x = \frac{1}{x+}\,\frac{x^2}{1-2x+}\,\frac{(1-x)^2}{2x\quad+}\,\frac{(1+x)^2}{1-2x\;+}\,\frac{(2-x)^2}{2x\quad+}\,\frac{(2+x)^2}{1-2x\;+}\cdots.}$$

Putting $x = 1/4$, give a new proof of Lord Brouncker's formula.

(c) Derive

$$\boxed{\frac{\tan\pi x}{\pi x} = 1 + \frac{x}{1-2x+}\,\frac{(1-x)^2}{2x\quad+}\,\frac{(1+x)^2}{1-2x\;+}\,\frac{(2-x)^2}{2x\quad+}\,\frac{(2+x)^2}{1-2x\;+}\cdots.}$$

7. Recall that $\frac{\pi}{\sin\pi x} = \frac{1}{x} + \sum_{n=1}^{\infty}\frac{2x}{n^2-x^2}$. From this, derive the beautiful expression

$$\boxed{\frac{\sin\pi x}{\pi x} = 1 - \frac{x}{1+}\,\frac{(1-x)^2}{2x\quad+}\,\frac{(1+x)^2}{1-2x\;+}\,\frac{(2-x)^2}{2x\quad+}\,\frac{(2+x)^2}{1-2x\;+}\cdots.}$$

Suggestion: First break up $\frac{2x}{n^2-x^2}$ and use an argument as you did for $\pi\cot\pi x$ to get a continued fraction expansion for $\frac{\pi}{\sin\pi x}$. From this, deduce the continued fraction expansion for $\sin\pi x/\pi x$.

8. From the expansion $\frac{\pi}{4\cos\frac{\pi x}{2}} = \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)}{(2n+1)^2-x^2}$ derive the beautiful expression

$$\boxed{\frac{\cos\frac{\pi x}{2}}{\frac{\pi}{2}} = x + 1 + \frac{(x+1)^2}{-2\cdot 1\;+}\,\frac{(x-1)^2}{-2\quad+}\,\frac{(x-3)^2}{2\cdot 3\quad+}\,\frac{(x+3)^2}{2\quad+}\,\frac{(x+5)^2}{-2\cdot 5\;+}\,\frac{(x-5)^2}{-2\quad+}\cdots.}$$

9. (Cf. [**89**]) In this problem we turn infinite products to continued fractions.

(a) Let $\alpha_1, \alpha_2, \alpha_3, \ldots$ be a sequence of real numbers with $\alpha_k \neq 0, -1$. Define sequences $b_1, b_2, b_3, \ldots$ and $a_0, a_1, a_2, \ldots$ by $b_1 = (1+\alpha_0)\alpha_1$, $a_0 = 1+\alpha_0$, $a_1 = 1$, and

$$b_n = -(1+\alpha_{n-1})\frac{\alpha_n}{\alpha_{n-1}} \quad , \quad \alpha_n = 1 - a_n \quad \text{for } n = 2, 3, 4, \ldots .$$

Prove (say by induction) that for any $n \in \mathbb{N}$,

$$\prod_{k=0}^{n}(1+\alpha_k) = a_0 + \frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\cdots\frac{b_n}{+\,a_n}.$$

Taking $n \to \infty$, we get a similar formula for infinite products and fractions.

(b) Using that $\frac{\sin\pi x}{\pi x} = \prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right) = (1-x)(1+x)\left(1-\frac{x}{2}\right)\left(1+\frac{x}{2}\right)\left(1-\frac{x}{3}\right)\left(1+\frac{x}{3}\right)\cdots$ and (a), derive the continued fraction expansion

$$\boxed{\frac{\sin\pi x}{\pi x} = 1 - \frac{x}{1+}\,\frac{1\cdot(1-x)}{x\quad+}\,\frac{1\cdot(1+x)}{1-x\;+}\,\frac{2\cdot(2-x)}{x\quad+}\,\frac{2\cdot(2+x)}{1-x\;+}\cdots.}$$

(c) Putting $x = 1/2$, prove (7.12). Putting $x = -1/2$, derive another continued fraction for $\pi/2$.

7.3. Recurrence relations, Diophantus' tomb, and shipwrecked sailors

In this section we define the Wallis-Euler recurrence relations, which generate sequences of numerators and denominators for convergents of continued fractions. Diophantine equations is the subject of finding integer solutions to polynomial equations. Continued fractions (through the special properties of the Wallis-Euler recurrence relations) turn out to play a very important role in this subject.

7.3.1. Convergents and recurrence relations. We shall call a continued fraction

$$a_0 + \frac{b_1}{a_1 +} \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \cdots \frac{b_n}{+ a_n +} \cdots \tag{7.15}$$

nonnegative if the a_n, b_n's are real numbers with $a_n > 0, b_n \geq 0$ for all $n \geq 1$ (a_0 can be any sign). We shall not spend a lot of time on continued fractions when the a_n's and b_n's in (7.15), for $n \geq 1$, are arbitrary real numbers; it is only for nonnegative infinite continued fractions that we develop their convergence properties in Section 7.4. However, we shall come across continued fractions where some of the a_n, b_n are negative — see for instance the beautiful expression (7.52) for $\cot x$ (and the following one for $\tan x$)! We focus on continued fractions with $a_n, b_n > 0$ for $n \geq 1$ in order to avoid some possible contradictory statements. For instance, the convergents of the elementary example $\frac{1}{1+} \frac{-1}{1} + \frac{1}{1}$ has some weird properties. Let us form its convergents: $c_1 = 1$, which is OK, but

$$c_2 = \frac{1}{1+} \frac{-1}{1} = \cfrac{1}{1 + \cfrac{-1}{1}} = \frac{1}{1-1} = \frac{1}{0} = ???,$$

which is not OK.[1] However,

$$c_3 = \frac{1}{1+} \frac{-1}{1} + \frac{1}{1} = \cfrac{1}{1 + \cfrac{-1}{1 + \cfrac{1}{1}}} = \cfrac{1}{1 + \cfrac{-1}{2}} = \cfrac{1}{\frac{1}{2}} = 2,$$

which is OK again! To avoid such craziness, we shall focus on continued fractions with $a_n > 0$ for $n \geq 1$ and $b_n \geq 0$, but *we emphasize that much of what we do in this section and the next works in greater generality.*

Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers with $a_n > 0, b_n \geq 0$ for all $n \geq 1$ (there is no restriction on a_0). The following sequences $\{p_n\}$, $\{q_n\}$ are central in the theory of continued fractions:

$$\boxed{\begin{array}{l} p_n = a_n p_{n-1} + b_n p_{n-2} \quad , \quad q_n = a_n q_{n-1} + b_n q_{n-2} \\ p_{-1} = 1 \ , \ p_0 = a_0 \quad , \quad q_{-1} = 0 \ , \ q_0 = 1. \end{array}} \tag{7.16}$$

We shall call these recurrence relations the **Wallis-Euler recurrence relations** ... you'll see why they're so central in a moment. In particular,

$$\boxed{\begin{array}{l} p_1 = a_1 p_0 + b_1 p_{-1} = a_1 a_0 + b_1 \\ q_1 = a_1 q_0 + b_1 q_{-1} = a_1. \end{array}} \tag{7.17}$$

We remark that $q_n > 0$ for $n = 0, 1, 2, 3, \ldots$. This is easily proved by induction: Certainly, $q_0 = 1, q_1 = a_1 > 0$ (recall that $a_n > 0$ for $n \geq 1$); thus assuming that $q_n > 0$ for $n = 0, \ldots, n-1$, we have (recall that $b_n \geq 0$),

$$q_n = a_n q_{n-1} + b_n q_{n-2} > 0 \cdot 0 + 0 = 0,$$

[1]Actually, in the continued fraction community, we always define $a/0 = \infty$ for $a \neq 0$ so we can get around this division by zero predicament by simply defining it away.

and our induction is complete. Observe that the zero-th convergent of the continued fraction (7.15) is $c_0 = a_0 = p_0/q_0$ and the first convergent is

$$c_1 = a_0 + \frac{b_1}{a_1} = \frac{a_1 a_0 + b_1}{a_1} = \frac{p_1}{q_1}.$$

The central property of the p_n, q_n's is the fact that $c_n = p_n/q_n$ for all n.

THEOREM 7.4. *For any positive real number x, we have*

$$\boxed{a_0 + \frac{b_1}{a_1 +}\,\frac{b_2}{a_2 +}\,\frac{b_3}{a_3 +}\,\cdots\,\frac{b_n}{+\;x} = \frac{x p_{n-1} + b_n p_{n-2}}{x q_{n-1} + b_n q_{n-2}}, \qquad n = 1, 2, 3, \ldots.} \tag{7.18}$$

(Note that the denominator is > 0 because $q_n > 0$ for $n \geq 0$.) In particular, setting $x = a_n$ and using the definition of p_n, q_n, we have

$$c_n = a_0 + \frac{b_1}{a_1 +}\,\frac{b_2}{a_2 +}\,\frac{b_3}{a_3 +}\,\cdots\,\frac{b_n}{+\;a_n} = \frac{p_n}{q_n}, \qquad n = 0, 1, 2, 3, \ldots.$$

PROOF. We prove (7.18) by induction on the number of terms after a_0. The proof for one term after a_0 is easy: $a_0 + \frac{b_1}{x} = \frac{a_0 x + b_1}{x} = \frac{x p_0 + b_1 p_{-1}}{x q_0 + q_{-1}}$, since $p_0 = a_0$, $p_{-1} = 1$, $q_0 = 1$, and $q_{-1} = 0$. Assume that (7.18) holds when there are n terms after a_0; we shall prove it holds for fractions with $n + 1$ terms after a_0. To do so, we write (see Problem 4 in Exercises 7.1 for the general technique)

$$a_0 + \frac{b_1}{a_1 +}\,\frac{b_2}{a_2 +}\,\cdots\,\frac{b_n}{+\;a_n +}\,\frac{b_{n+1}}{x} = a_0 + \frac{b_1}{a_1 +}\,\frac{b_2}{a_2 +}\,\cdots\,\frac{b_n}{+\;y},$$

where

$$y := a_n + \frac{b_{n+1}}{x} = \frac{x a_n + b_{n+1}}{x}.$$

Therefore by our induction hypothesis, we have

$$\begin{aligned}
a_0 + \frac{b_1}{a_1 +}\,\frac{b_2}{a_2 +}\,\cdots\,\frac{b_{n+1}}{+\;x} = \frac{y p_{n-1} + b_n p_{n-2}}{y q_{n-1} + b_n q_{n-2}} &= \frac{\left(\dfrac{x a_n + b_{n+1}}{x}\right) p_{n-1} + b_n p_{n-2}}{\left(\dfrac{x a_n + b_{n+1}}{x}\right) q_{n-1} + b_n q_{n-2}} \\
&= \frac{x a_n p_{n-1} + b_{n+1} p_{n-1} + x b_n p_{n-2}}{x a_n q_{n-1} + b_{n+1} q_{n-1} + x b_n q_{n-2}} \\
&= \frac{x(a_n p_{n-1} + b_n p_{n-2}) + b_{n+1} p_{n-1}}{x(a_n q_{n-1} + b_n q_{n-2}) + b_{n+1} q_{n-1}} \\
&= \frac{x p_n + b_{n+1} p_{n-1}}{x q_n + b_{n+1} q_{n-1}},
\end{aligned}$$

which completes our induction step and finishes our proof. □

In the next theorem, we give various useful identities that the p_n, q_n satisfy.

THEOREM 7.5 (**Fundamental recurrence relations**). *For all $n \geq 1$, the following identities hold:*

$$\boxed{\begin{aligned} p_n q_{n-1} - p_{n-1} q_n &= (-1)^{n-1} b_1 b_2 \cdots b_n \\ p_n q_{n-2} - p_{n-2} q_n &= (-1)^n a_n b_1 b_2 \cdots b_{n-1} \end{aligned}}$$

and (where the formula for $c_n - c_{n-2}$ is only valid for $n \geq 2$)

$$\boxed{c_n - c_{n-1} = \frac{(-1)^{n-1} b_1 b_2 \cdots b_n}{q_n\, q_{n-1}} \quad , \quad c_n - c_{n-2} = \frac{(-1)^n a_n b_1 b_2 \cdots b_{n-1}}{q_n\, q_{n-2}}.}$$

PROOF. To prove that $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} b_1 b_2 \cdots b_n$ for $n = 1, 2, \ldots$, we proceed by induction. For $n = 1$, the left-hand side is (see (7.17))

$$p_1 q_0 - p_0 q_1 = (a_1 a_0 + b_1) \cdot 1 - a_0 \cdot a_1 = b_1,$$

which is the right-hand side when $n = 1$. Assume our equality holds for n; we prove it holds for $n + 1$. By our Wallis-Euler recurrence relations, we have

$$\begin{aligned} p_{n+1} q_n - p_n q_{n+1} &= \big(a_{n+1} p_n + b_{n+1} p_{n-1}\big) q_n - p_n \big(a_{n+1} q_n + b_{n+1} q_{n-1}\big) \\ &= b_{n+1} p_{n-1} q_n - p_n b_{n+1} q_{n-1} \\ &= -b_{n+1} \big(p_n q_{n-1} - p_{n-1} q_n\big) \\ &= -b_{n+1} \cdot (-1)^{n-1} b_1 b_2 \cdots b_n = (-1)^n b_1 b_2 \cdots b_n b_{n+1}, \end{aligned}$$

which completes our induction step. To prove the second equality, we use the Wallis-Euler recurrence relations and the equality just proved:

$$\begin{aligned} p_n q_{n-2} - p_{n-2} q_n &= \big(a_n p_{n-1} + b_n p_{n-2}\big) q_{n-2} - p_{n-2} \big(a_n q_{n-1} + b_n q_{n-2}\big) \\ &= a_n p_{n-1} q_{n-2} - p_{n-2} a_n q_{n-1} \\ &= a_n \big(p_{n-1} q_{n-2} - p_{n-2} q_{n-1}\big) \\ &= a_n \cdot (-1)^{n-2} b_1 b_2 \cdots b_{n-1} = (-1)^n a_n b_1 b_2 \cdots b_{n-1}. \end{aligned}$$

Finally, the equations for $c_n - c_{n-1}$ and $c_n - c_{n-2}$ follow from dividing

$$\begin{aligned} p_n q_{n-1} - p_{n-1} q_n &= (-1)^{n-1} b_1 b_2 \cdots b_n \\ p_n q_{n-2} - p_{n-2} q_n &= (-1)^{n-1} a_n b_1 b_2 \cdots b_{n-1} \end{aligned}$$

by $q_n\, q_{n-1}$ and $q_n\, q_{n-2}$, respectively. □

For simple continued fractions, the Wallis-Euler relations (7.16) and (7.17) and the fundamental recurrence relations take the following particularly simple forms:

COROLLARY 7.6 (**Simple fundamental recurrence relations**). *For simple continued fractions, for all $n \geq 1$, if*

$$\boxed{\begin{array}{lcl} p_n = a_n p_{n-1} + p_{n-2} & , & q_n = a_n q_{n-1} + q_{n-2} \\ p_0 = a_0 \ , \ p_1 = a_0 a_1 + 1 & , & q_0 = 1 \ , \ q_1 = a_1, \end{array}}$$

then $c_n = p_n / q_n$ for all $n \geq 0$, and for any $x > 0$,

$$\boxed{\langle a_0; a_1, a_2, a_3, \ldots, a_n, x\rangle = \frac{x p_{n-1} + p_{n-2}}{x q_{n-1} + q_{n-2}}, \quad n = 1, 2, 3, \ldots.} \tag{7.19}$$

Moreover, for all $n \geq 1$, the following identities hold:

$$\boxed{\begin{aligned} p_n q_{n-1} - p_{n-1} q_n &= (-1)^{n-1} \\ p_n q_{n-2} - p_{n-2} q_n &= (-1)^n a_n \end{aligned}}$$

and

$$\boxed{c_n - c_{n-1} = \frac{(-1)^{n-1}}{q_n\, q_{n-1}} \quad , \quad c_n - c_{n-2} = \frac{(-1)^n a_n}{q_n\, q_{n-2}},}$$

where $c_n - c_{n-2}$ *is only valid for* $n \geq 2$.

We also have the following interesting result.

COROLLARY 7.7. *All the* p_n, q_n *for a simple continued fraction are relatively prime; that is,* $c_n = p_n/q_n$ *is automatically in lowest terms.*

PROOF. The fact that p_n, q_n are in lowest terms follows from the fact that

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1},$$

so if an integer happens to divide divide both p_n and q_n, then it divides $p_n q_{n-1} - p_{n-1} q_n$ also, so it must divide $(-1)^{n-1} = \pm 1$ which is impossible unless the number was ± 1. □

7.3.2. ★ Diophantine equations and sailors, coconuts, and monkeys. From Section 7.1, we know that any rational number can be written as a finite simple continued fraction. Also, any finite simple continued fraction is certainly a rational number because it is made up of additions and divisions of rational numbers and the rational numbers are closed under such operations. (For proofs of these statements see Problem 2 in Exercises 7.1.) Now as we showed at the beginning of Section 7.1, we can write

$$\frac{157}{68} = 2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{5}}} = \langle 2; 3, 4, 5 \rangle,$$

which has an odd number of terms after the integer part 2. Observe that we can write this in another way:

$$\frac{157}{68} = 2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{1}}}} = \langle 2; 3, 4, 4, 1 \rangle,$$

which has an even number of terms after the integer part. This example is typical: Any finite simple continued fraction can be written with an even or odd number of terms (by modifying the last term by 1). We summarize these remarks in the following theorem, which we shall use in Theorem 7.9.

THEOREM 7.8. *A real number can be expressed as a finite simple continued fraction if and only if it is rational, in which case, the rational number can be expressed as a continued fraction with either an even or an odd number of terms.*

The proof of this theorem shall be left to you. We now come to the subject of diophantine equations, which are polynomial equations we wish to find integer solutions. We shall study very elementary diophantine equations in this section, the linear ones. Before doing so, it might of interest to know that diophantine equations is named after Diophantus of Alexandrea, a Greek mathematician who lived around 250 A.D. He is famous for at least two things: His book *Arithmetica*, which studies equations that we now call diophantine equations in his honor, and for the following riddle, which was supposedly written on his tombstone:

> *This tomb hold Diophantus Ah, what a marvel! And the tomb tells scientifically the measure of his life. God vouchsafed that he should be a boy for the sixth part of his life; when a twelfth was added, his cheeks acquired a beard; He kindled for him the light of marriage after a seventh, and in the fifth year after his marriage He granted him a son. Alas! late-begotten and miserable child, when he had reached the measure of half his father's life, the chill grave took him. After consoling his grief by this science of numbers for four years, he reached the end of his life.* [**124**].

Try to find how old Diophantus was when he died using elementary algebra. (Let $x =$ his age when he died; then you should end up with trying to solve the equation $x = \frac{1}{6}x + \frac{1}{12}x + \frac{1}{7}x + 5 + \frac{1}{2}x + 4$, obtaining $x = 84$.) Here is an easy way to find his age: Unravelling the above fancy language, and picking out two facts, we know that 1/12-th of his life was in youth and 1/7-th was as a bachelor. In particular, his age must divide 7 and 12. The only integer that does this, and which is within the human lifespan, is $7 \cdot 12 = 84$. In particular, he spent $84/6 = 14$ years as a child, $84/12 = 7$ as a youth, $84/7 = 12$ years as a bachelor. He married at $14 + 7 + 12 = 33$, at $33 + 5 = 38$, his son was born, who later died at the age of $84/2 = 42$ years old, when Diophantus was 80. Finally, after 4 years doing the "science of numbers", Diophantus died at the ripe old age of 84.

After taking a moment to wipe away our tears, let us consider the following.

THEOREM 7.9. *If $a, b \in \mathbb{N}$ are relatively prime, then for any $c \in \mathbb{Z}$, the equation*

$$ax - by = c$$

has an infinite number of integer solutions (x, y). Moreover, if (x_0, y_0) is any one integral solution of the equation with $c = 1$, then for general $c \in \mathbb{Z}$, all solutions are of the form

$$x = cx_0 + bt \quad , \quad y = cy_0 + at \quad , \quad t \in \mathbb{Z}.$$

PROOF. In Problem 7 we ask you to prove this theorem using Problem 5 in Exercises 2.4; but we shall use continued fractions just for fun. We first solve the equation $ax - by = 1$. To do so, we write a/b as a finite simple continued fraction: $a/b = \langle a_0; a_1, a_2, \ldots, a_n \rangle$ and by Theorem 7.8 we can choose n to be *odd*. Then a/b is equal to the n-th convergent p_n/q_n, which implies that $p_n = a$ and $q_n = b$. Also, by our relations in Corollary 7.6, we know that

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1} = 1,$$

where we used that n is odd. Since p_n and q_n are relatively prime and $a/b = p_n/q_n$, we must have $p_n = a$ and $q_n = b$. Therefore, $aq_{n-1} - bp_{n-1} = 1$, so

$$\boxed{(x_0, y_0) = (q_{n-1}, p_{n-1})} \tag{7.20}$$

solves $ax_0 - by_0 = 1$. Multiplying $ax_0 - by_0 = 1$ by c we get

$$a(cx_0) - b(cy_0) = c.$$

Then $ax - by = c$ holds if and only if (by replacing c with $a(cx_0) - b(cy_0)$)

$$a(x - cx_0) = b(y - cy_0).$$

This shows that a divides $b(y - cy_0)$, which can be possible if and only if a divides $y - cy_0$ since a and b are relatively prime. Thus, $y - cy_0 = at$ for some $t \in \mathbb{Z}$. Plugging $y - cy_0 = at$ into the equation $a(x - cx_0) = b(y - cy_0)$, we get

$$a(x - cx_0) = b \cdot (at) = abt.$$

Cancelling a, we get $x - cx_0 = bt$ and our proof is now complete. □

We remark that we need a and b need to be relatively prime; for example, the equation $2x - 4y = 1$ has no integer solutions (because the left hand side is always even, so can never equal 1). We also remark that the proof of Theorem 7.9; in particular, the formula (7.20), also shows us *how* to find (x_0, y_0): Just write a/b as a simple continued fraction with an *odd* number n terms after the integer part of a/b and compute the $(n-1)$-st convergent to get $(x_0, y_0) = (q_{n-1}, p_{n-1})$.

Example 7.10. Consider the diophantine equation

$$157x - 68y = 12.$$

We already know that the continued fraction expansion of $a/b = \frac{157}{68}$ with an odd $n = 3$ is $\frac{157}{68} = \langle 2; 3, 4, 5 \rangle = \langle a_0; a_1, a_2, a_3 \rangle$. Thus,

$$c_2 = 2 + \cfrac{1}{3 + \cfrac{1}{4}} = 2 + \frac{4}{13} = \frac{30}{13}.$$

Therefore, $(13, 30)$ is one solution of $157x - 68y = 1$, which we should check just to be sure:

$$157 \cdot 13 - 68 \cdot 30 = 2041 - 2040 = 1.$$

Since $cx_0 = 12 \cdot 13 = 156$ and $cy_0 = 12 \cdot 30 = 360$, the general solution of $157x - 68y = 12$ is

$$x = 156 + 68t \quad , \quad y = 360 + 157t, \qquad t \in \mathbb{Z}.$$

Example 7.11. We now come to a fun puzzle that involves diophantine equations; for more cool coconut puzzles, see [**62, 63**], [**173**], [**163**], and Problem 5. Five sailors get shipwrecked on an island where there is only a coconut tree and a very slim monkey. The sailors gathered all the coconuts into a gigantic pile and went to sleep. At midnight, one sailor woke up, and because he didn't trust his mates, he divided the coconuts into five equal piles, but with one coconut left over. He throws the extra one to the monkey, hides his pile, puts the remaining coconuts back into a pile, and goes to sleep. At one o'clock, the second sailor woke up, and because he was untrusting of his mates, he divided the coconuts into five equal piles, but again with one coconut left over. He throws the extra one to the monkey, hides his pile, puts the remaining coconuts back into a pile, and goes to sleep. This exact same scenario continues throughout the night with the other three sailors. In the morning, all the sailors woke up, pretending as if nothing happened and divided the now minuscule pile of coconuts into five equal piles, and they find yet again one coconut left over, which they throw to the now very overweight monkey. Question: What is the smallest possible number of coconuts in the original pile?

Let $x =$ the original number of coconuts. Remember that sailor #1 divided x into five parts, but with one left over. This means that if y_1 is the number that he took, then $x = 5y_1 + 1$ and he left $4 \cdot y_1$ coconuts. In other words, he took

$$\frac{1}{5}(x - 1) \text{ coconuts, leaving } 4 \cdot \frac{1}{5}(x - 1) = \frac{4}{5}(x - 1) \text{ coconuts.}$$

Similarly, if y_2 is the number of coconuts that the sailor #2 took, then $\frac{4}{5}(x-1) = 5y_2 + 1$ and he left $4 \cdot y_2$ coconuts. That is, the second sailor took

$$\frac{1}{5}\cdot\left(\frac{4}{5}(x-1)-1\right) = \frac{4x-9}{25} \text{ coconuts, leaving } 4\cdot\frac{4x-9}{25} = \frac{16x-36}{25} \text{ coconuts.}$$

Repeating this argument, we find that sailors #3, #4, and #5 left

$$\frac{64x-244}{125}\quad,\quad\frac{256x-1476}{625}\quad,\quad\frac{1024x-8404}{3125}$$

coconuts, respectively. At the end, the sailors divided this last amount of coconuts into five piles, with one coconut left over. Thus, if y is the number of coconuts in each pile, then we must have

$$\frac{1024x-8404}{3125} = 5y+1 \quad\Longrightarrow\quad 1024x - 15625y = 11529.$$

The equation $1024x - 15625y = 11529$ is just a diophantine equation since we want *integers* x, y solving this equation. Moreover, $1024 = 2^{10}$ and $15625 = 5^6$ are relatively prime, so we can solve this equation by Theorem 7.9. First, we solve $1024x - 15625y = 1$, which we solve by writing $1024/15625$ as a continued fraction (this takes some algebra) and forcing n to be odd (in this case $n = 9$):

$$\frac{1024}{15625} = \langle 0; 15, 3, 1, 6, 2, 1, 3, 2, 1\rangle.$$

Second, we take the $(n-1)$-st convergent:

$$c_8 = \langle 0; 15, 3, 1, 6, 2, 1, 3, 2\rangle = \frac{711}{10849}.$$

Thus, $(x_0, y_0) = (10849, 711)$. Since $cx_0 = 11529 \cdot 10849 = 125078121$ and $cy_0 = 11529 \cdot 711 = 8197119$, the integer solutions to $1024x - 15625y = 11529$ are

$$x = 125078121 + 15625t \quad,\quad y = 8197119 + 1024t \quad,\quad t \in \mathbb{Z}. \tag{7.21}$$

This of course gives us infinitely many solutions. However, we want the smallest *nonnegative* solutions since x and y represent numbers of coconuts; thus, we need

$$x = 125078121 + 15625t \geq 0 \quad\Longrightarrow\quad t \geq -\frac{125078121}{15625} = -8004.999744\ldots,$$

and

$$y = 8197119 + 1024t \geq 0 \quad\Longrightarrow\quad t \geq -\frac{8197119}{1024} = -8004.9990234\ldots.$$

Thus, taking $t = -8004$ in (7.21), we finally arrive at $x = 15621$ and $y = 1023$. In conclusion, the smallest number of coconuts in the original piles is 15621 coconuts. By the way, you can solve this coconut problem *without* continued fractions using nothing more than basic high school algebra; try it!

Exercises 7.3.

1. Find the general integral solutions of

$$(a)\ 7x - 11y = 1 \quad,\quad (b)\ 13x - 3y = 5 \quad,\quad (c)\ 13x - 15y = 100.$$

2. If all the $a_0, a_2, \ldots, a_n > 0$ (which guarantees that $p_0 = a_0 > 0$), prove that

$$\frac{p_n}{p_{n-1}} = \langle a_n; a_{n-1}, a_{n-2}, \ldots, a_2, a_1, a_0\rangle \quad\text{and}\quad \frac{q_n}{q_{n-1}} = \langle a_n; a_{n-1}, a_{n-2}, \ldots, a_2, a_1\rangle$$

for $n = 1, 2, \ldots$. Suggestion: Observe that $\frac{p_k}{p_{k-1}} = \frac{a_k p_{k-1} + p_{k-2}}{p_{k-1}} = a_k + \frac{1}{\frac{p_{k-1}}{p_{k-2}}}$.

3. In this problem, we relate the Fibonacci numbers to continued fractions. Recall that the Fibonacci sequence $\{F_n\}$ is defined as $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Let $p_n/q_n = \langle a_0; a_1, \ldots, a_n\rangle$ where all the a_k's are equal to 1.
   (a) Prove that $p_n = F_{n+2}$ and $q_n = F_{n+1}$ for all $n = -1, 0, 1, 2, \ldots$. Suggestion: Use the Wallis-Euler recurrence relations.
   (b) From facts known about convergents, prove that F_n and F_{n+1} are relatively prime and derive the following famous identity:
$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n \qquad \textbf{(Cassini or Simpson's identity)}.$$
4. Imitating the proof of Theorem 7.9, show that a solution of $ax - by = -1$ can be found by writing a/b as a simple continued fraction with an *even* number n terms after the integer part of a/b and finding the $(n-1)$-th convergent. Apply this method to find a solution of $157x - 68y = -1$.
5. (**Coconut problems**) Here are some more coconut problems:
   (a) Solve the coconut problem assuming the same antics as in the text, except for one thing: there are no coconuts left over for the monkey at the end. That is, what is the smallest possible number of coconuts in the original pile given that after the sailors divided the coconuts in the morning, there are no coconuts left over?
   (b) Solve the coconut problem assuming the same antics as in the text except that during the night each sailor divided the pile into five equal piles with *none* left over; however, after he puts the remaining coconuts back into a pile, the monkey (being a thief himself) steals one coconut from the pile (before the next sailor wakes up). In the morning, there is still one coconut left over for the monkey.
   (c) Solve the coconut problem when there are only three sailors to begin with, otherwise everything is the same as in the text (e.g. one coconut left over at the end). Solve this same coconut problem when there are no coconuts left over at the end.
   (d) Solve the coconut problem when there are seven sailors, otherwise everything is the same as in the text. (Warning: Set aside an evening for long computations!)
6. Let $\alpha = \langle a_0; a_1, a_2, \ldots, a_m\rangle$, $\beta = \langle b_0; b_1, \ldots, b_n\rangle$ with $m, n \geq 0$ and the a_k, b_k's integers with $a_m, b_n > 1$ (such finite continued fractions are called **regular**). Prove that if $\alpha = \beta$, then $a_k = b_k$ for all $k = 0, 1, 2, \ldots$. In other words, distinct regular finite simple continued fractions define different rational numbers.
7. Prove Theorem 7.9 using Problem 5 in Exercises 2.4.

7.4. Convergence theorems for infinite continued fractions

Certainly the continued fraction $\langle 1; 1, 1, 1, 1, \ldots\rangle$ (if it converges), should be a very special number — it is, it turns out to be the golden ratio! In this section we shall investigate the delicate issues surrounding convergence of infinite continued fractions (see Theorem 7.14, the continued fraction convergence theorem); in particular, we prove that *any* simple continued fraction converges. We also show how to write *any* real number as a simple continued fraction via the **canonical continued fraction algorithm**. Finally, we prove that a real number is irrational if and only if its simple continued fraction expansion is infinite.

7.4.1. Monotonicity properties of convergents. Consider a nonnegative infinite continued fraction
$$a_0 + \frac{b_1}{a_1 +}\,\frac{b_2}{a_2 +}\,\frac{b_3}{a_3 +}\,\cdots\,\frac{b_n}{+\,a_n +}\,\cdots,$$
where recall that nonnegative means the a_n, b_n's are real numbers with $a_n > 0, b_n \geq 0$ for all $n \geq 1$, and there is no restriction on a_0. The Wallis-Euler recurrence

relations (7.16) are

$$p_n = a_n p_{n-1} + b_n p_{n-2} \quad , \quad q_n = a_n q_{n-1} + b_n q_{n-2}$$
$$p_{-1} = 1 \ , \ p_0 = a_0 \quad , \quad q_{-1} = 0 \ , \ q_0 = 1.$$

Then (cf. (7.17))

$$p_1 = a_1 p_0 + b_1 p_{-1} = a_1 a_0 + b_1 \quad , \quad q_1 = a_1 q_0 + b_1 q_{-1} = a_1,$$

and all the q_n's are positive (see discussion below (7.17)). By Theorem 7.4 we have $c_n = p_n/q_n$ for all n and by Theorem 7.5, for all $n \geq 1$ the fundamental recurrence relations are

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} b_1 b_2 \cdots b_n$$
$$p_n q_{n-2} - p_{n-2} q_n = (-1)^n a_n b_1 b_2 \cdots b_{n-1}$$

and

$$c_n - c_{n-1} = \frac{(-1)^{n-1} b_1 b_2 \cdots b_n}{q_n \, q_{n-1}} \quad , \quad c_n - c_{n-2} = \frac{(-1)^n a_n b_1 b_2 \cdots b_{n-1}}{q_n \, q_{n-2}},$$

where $c_n - c_{n-2}$ is only valid for $n \geq 2$. Using these fundamental recurrence relations, we shall prove the following monotonicity properties of the c_n's, which is important in the study of the convergence properties of the c_n's.

THEOREM 7.10. *Assume that* $b_n > 0$ *for each* n. *Then the convergents* $\{c_n\}$ *satisfy the inequalities: For all* $n \in \mathbb{N}$,

$$c_0 < c_2 < c_4 < \cdots < c_{2n} < c_{2n-1} < \cdots < c_5 < c_3 < c_1.$$

That is, the even convergents form a strictly increasing sequence while the odd convergent form a strictly decreasing sequence.

PROOF. Replacing n with $2n$ in the fundamental recurrence relation for $c_n - c_{n-2}$, we see that

$$c_{2n} - c_{2n-2} = \frac{(-1)^{2n-2} a_{2n} b_1 b_2 \cdots b_{2n-1}}{q_{2n} \, q_{2n-1}} = \frac{a_{2n} b_1 b_2 \cdots b_{2n-1}}{q_{2n} \, q_{2n-1}} > 0.$$

This shows that $c_{2n-2} < c_{2n}$ for all $n \geq 1$ and hence, $c_0 < c_2 < c_4 < \cdots$. Replacing n with $2n-1$ in the fundamental relation for $c_n - c_{n-2}$ can be used to prove that the odd convergents form a strictly decreasing sequence. Replacing n with $2n$ in the fundamental recurrence relation for $c_n - c_{n-1}$, we see that

$$(7.22) \quad c_{2n} - c_{2n-1} = \frac{(-1)^{2n-1} b_1 b_2 \cdots b_{2n}}{q_{2n} \, q_{2n-1}} = -\frac{b_1 b_2 \cdots b_{2n}}{q_{2n} \, q_{2n-1}} < 0 \implies c_{2n} < c_{2n-1}.$$

□

If the continued fraction is actually finite; that is, if $b_{\ell+1} = 0$ for some ℓ, then this theorem still holds, but we need to make sure that $2n \leq \ell$. By the monotone criterion for sequences, we have

COROLLARY 7.11. *The limits of the even and odd convergents exist, and*

$$c_0 < c_2 < c_4 < \cdots < \lim c_{2n} \leq \lim c_{2n-1} < \cdots < c_5 < c_3 < c_1.$$

7.4.2. Convergence results for continued fractions. As a consequence of the previous corollary, it follows that $\lim c_n$ exists if and only if $\lim c_{2n} = \lim c_{2n-1}$, which holds if and only if

$$c_{2n} - c_{2n-1} = \frac{-b_1 b_2 \cdots b_{2n}}{q_{2n}\, q_{2n-1}} \to 0 \quad \text{as } n \to \infty. \tag{7.23}$$

In the following theorem, we give one condition under which this is satisfied.

THEOREM 7.12. *Let* $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=1}^{\infty}$ *be sequences such that* $a_n, b_n > 0$ *for* $n \geq 1$ *and*

$$\sum_{n=1}^{\infty} \frac{a_n a_{n+1}}{b_{n+1}} = \infty.$$

Then (7.23) *holds, so the continued fraction* $\xi := a_0 + \frac{b_1}{a_1+}\frac{b_2}{a_2+}\frac{b_3}{a_3+}\frac{b_4}{a_4+}\cdots$ *converges. Moreover, for any even* j *and odd* k*, we have*

$$c_0 < c_2 < c_4 < \cdots < c_j < \cdots < \xi < \cdots < c_k < \cdots < c_5 < c_3 < c_1.$$

PROOF. Observe that for any $n \geq 1$, we have $q_n = a_n q_{n-1} + b_n q_{n-2} \geq a_n q_{n-1}$ since $b_n, q_{n-2} \geq 0$. Replacing n by $n-1$, for $n \geq 2$ we have

$$q_n = a_n q_{n-1} + b_n q_{n-2} \geq a_n \cdot (a_{n-1} q_{n-2}) + b_n q_{n-2} = q_{n-2}(a_n a_{n-1} + b_n),$$

so

$$q_n \geq q_{n-2}(a_n a_{n-1} + b_n).$$

Applying this formula over and over again, we find that for any $n \geq 1$,

$$\begin{aligned} q_{2n} &\geq q_{2n-2}(a_{2n}a_{2n-1} + b_{2n}) \\ &\geq q_{2n-4}(a_{2n-2}a_{2n-3} + b_{2n-2}) \cdot (a_{2n}a_{2n-1} + b_{2n}) \\ &\geq \ \vdots \\ &\geq q_0(a_2 a_1 + b_2)(a_4 a_3 + b_4) \cdots (a_{2n}a_{2n-1} + b_{2n}). \end{aligned}$$

A similar argument shows that for any $n \geq 2$,

$$q_{2n-1} \geq q_1(a_3 a_2 + b_3)(a_5 a_4 + b_5) \cdots (a_{2n-1}a_{2n-2} + b_{2n-1}).$$

Thus, for any $n \geq 2$, we have

$$q_{2n}q_{2n-1} \geq q_0 q_1 (a_2 a_1 + b_2)(a_3 a_2 + b_3) \cdots (a_{2n-1}a_{2n-2} + b_{2n-1})(a_{2n}a_{2n-1} + b_{2n}).$$

Factoring out all the b_k's we conclude that

$$q_{2n}q_{2n-1} \geq q_0 q_1 b_2 \cdots b_{2n} \cdots \left(1 + \frac{a_2 a_1}{b_2}\right)\left(1 + \frac{a_3 a_2}{b_3}\right) \cdots \left(1 + \frac{a_{2n}a_{2n-1}}{b_{2n}}\right),$$

which shows that

$$\frac{b_1 b_2 \cdots b_{2n}}{q_{2n}\, q_{2n-1}} \leq \frac{b_1}{q_0 q_1} \cdot \frac{1}{\prod_{k=1}^{2n-1}\left(1 + \frac{a_k a_{k+1}}{b_{k+1}}\right)}. \tag{7.24}$$

Now recall that (see Theorem 6.2) a series $\sum_{k=1}^{\infty} \alpha_k$ of positive real numbers converges if and only if the infinite product $\prod_{k=1}^{\infty}(1 + \alpha_k)$ converges. Thus, since we are given that $\sum_{k=1}^{\infty} \frac{a_k a_{k+1}}{b_{k+1}} = \infty$, we have $\prod_{k=1}^{\infty}\left(1 + \frac{a_k a_{k+1}}{b_{k+1}}\right) = \infty$ as well, so the right-hand side of (7.24) vanishes as $n \to \infty$. The fact that for even j and odd k, we have $c_0 < c_2 < c_4 < \cdots < c_j < \cdots < \xi < \cdots < c_k < \cdots < c_5 < c_3 < c_1$ follows from Corollary 7.11. This completes our proof. □

For another convergence theorem, see Problems 6 and 9. An important example for which this theorem applies is to simple continued fractions: For a simple continued fraction $\langle a_0; a_1, a_2, a_3, \ldots\rangle$, all the b_n's equal 1, so

$$\sum_{n=1}^{\infty} \frac{a_n a_{n+1}}{b_{n+1}} = \sum_{n=1}^{\infty} a_n a_{n+1} = \infty,$$

since all the a_n's are positive integers. Thus,

Corollary 7.13. *Infinite simple continued fractions always converge and if ξ is the limit of such a fraction, then the convergents $\{c_n\}$ satisfy*

$$c_0 < c_2 < c_4 < \cdots < c_{2n} < \cdots < \xi < \cdots < c_{2n-1} < c_5 < c_3 < c_1.$$

Example 7.12. In particular, the very special fraction $\Phi := \langle 1; 1, 1, 1, \ldots\rangle$ converges. To what you ask? Observe that

$$\Phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}} = 1 + \frac{1}{\Phi} \quad \Longrightarrow \quad \Phi = 1 + \frac{1}{\Phi}.$$

We can also get this formula from convergents: The n-th convergent of Φ is

$$c_n = 1 + \cfrac{1}{1 + \cfrac{1}{\ddots \, 1 + \cfrac{1}{1 + \cfrac{1}{1}}}} = 1 + \frac{1}{c_{n-1}}.$$

Thus, if we set $\Phi = \lim c_n$, which we know exists, then taking $n \to \infty$ on both sides of $c_n = 1 + \frac{1}{c_{n-1}}$, we get $\Phi = 1 + 1/\Phi$ just as before. Thus, $\Phi^2 - \Phi - 1 = 0$, which after solving for Φ we get

$$\Phi = \frac{1 + \sqrt{5}}{2},$$

the golden ratio.

As a unrelated side note, we remark that Φ can be used to get a fairly accurate (and well-known) approximation to π:

$$\boxed{\pi \approx \frac{6}{5}\Phi^2 = 3.1416\ldots.}$$

Example 7.13. The continued fraction $\xi := 3 + \frac{4}{6} \, {}_{+} \, \frac{4}{6} \, {}_{+} \, \frac{4}{6} \, {}_{+} \, \frac{4}{6} \cdots$ was studied by Rafael Bombelli (1526–1572) and was one of the first continued fractions ever to be studied. Since $\sum_{n=1}^{\infty} \frac{a_n a_{n+1}}{b_{n+1}} = \sum_{n=1}^{\infty} \frac{6^2}{4} = \infty$, this continued fraction converges. By the same reasoning, the continued fraction $\eta := 6 + \frac{4}{6} \, {}_{+} \, \frac{4}{6} \, {}_{+} \, \frac{4}{6} \cdots$ also converges. Moreover, $\xi = \eta - 3$ and

$$\eta = 6 + \cfrac{4}{6 + \cfrac{4}{6 + \cfrac{4}{\ddots}}} = 6 + \frac{1}{\eta} \quad \Longrightarrow \quad \eta = 6 + \frac{1}{\eta} \quad \Longrightarrow \quad \eta^2 - 6\eta - 1 = 0.$$

Solving this quadratic equation for ξ, we find that $\eta = 3+\sqrt{13}$. Hence, $\xi = \eta - 3 = \sqrt{13}$. Isn't this fun!

7.4.3. The canonical continued fraction algorithm and the continued fraction convergence theorem. Now what if we want to *construct* the continued fraction expansion of a real number? We already know how to construct such an expansion for rational numbers, so let us review this quickly; the same method will work for irrational numbers. Consider again our friend $\frac{157}{68} = \langle 2; 3, 4, 5\rangle$, and let us recall how we found its continued fraction expansion. First, we wrote $\xi_0 := \frac{157}{68}$ as

$$\xi_0 = 2 + \frac{1}{\xi_1} \quad , \quad \text{where } \xi_1 = \frac{68}{21} > 1.$$

In particular,

$$a_0 = 2 = \lfloor \xi_0 \rfloor,$$

where recall that $\lfloor x \rfloor$, where x is a real number, denotes the largest integer $\leq x$. Second, we wrote $\xi_1 = \frac{68}{21}$ as

$$\xi_1 = 3 + \frac{1}{\xi_2} \quad , \quad \text{where } \xi_2 = \frac{21}{5} > 1.$$

In particular,

$$a_1 = 3 = \lfloor \xi_1 \rfloor.$$

Third, we wrote

$$\xi_2 = \frac{21}{5} = 4 + \frac{1}{\xi_3} \quad , \quad \text{where } \xi_3 = 5 > 1.$$

In particular,

$$a_2 = 4 = \lfloor \xi_2 \rfloor.$$

Finally, $a_3 = \lfloor \xi_3 \rfloor = \xi_3$ cannot be broken up any further so we *stop here.* Hence,

$$\frac{157}{68} = \xi_0 = 2 + \frac{1}{\xi_1} = 2 + \cfrac{1}{3 + \cfrac{1}{\xi_2}} = 2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{\xi_3}}} = 2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{5}}}.$$

We've just found the **canonical (simple) continued fraction** of 157/68. Notice that we end with the number 5, which is greater than 1; this will always happen whenever we do the above procedure for a noninteger rational number (such continued fractions were called **regular** in Problem 6 of Exercises 7.3). We can do the same exact procedure for irrational numbers! Let ξ be an irrational number. First, we set $\xi_0 = \xi$ and define $a_0 := \lfloor \xi_0 \rfloor \in \mathbb{Z}$. Then, $0 < \xi_0 - a_0 < 1$ (note that $\xi_0 \neq a_0$ since ξ_0 is irrational), so we can write

$$\xi_0 = a_0 + \frac{1}{\xi_1} \quad , \quad \text{where } \xi_1 := \frac{1}{\xi_0 - a_0} > 1,$$

where we used that $0 < \xi_0 - a_0$. Note that ξ_1 is irrational because if not, then ξ_0 would be rational contrary to assumption. Second, we define $a_1 := \lfloor \xi_1 \rfloor \in \mathbb{N}$. Then, $0 < \xi_1 - a_1 < 1$, so we can write

$$\xi_1 = a_1 + \frac{1}{\xi_2} \quad , \quad \text{where } \xi_2 := \frac{1}{\xi_1 - a_1} > 1.$$

Note that ξ_2 is irrational. Third, we define $a_2 := \lfloor \xi_2 \rfloor \in \mathbb{N}$. Then, $0 < \xi_2 - a_2 < 1$, so we can write

$$\xi_2 = a_2 + \frac{1}{\xi_3} \quad , \quad \text{where } \xi_3 := \frac{1}{\xi_2 - a_2} > 1.$$

Note that ξ_3 is irrational. We can continue this procedure to "infinity" creating a sequence $\{\xi_n\}_{n=0}^{\infty}$ of real numbers with $\xi_n > 0$ for $n \geq 1$ called the **complete quotients** of ξ, and a sequence $\{a_n\}_{n=0}^{\infty}$ of integers with $a_n > 0$ for $n \geq 1$ called the **partial quotients** of ξ, such that

$$\xi_n = a_n + \frac{1}{\xi_{n+1}}, \quad n = 0, 1, 2, 3, \ldots.$$

Thus,

$$\xi = \xi_0 = a_0 + \frac{1}{\xi_1} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\xi_2}} = \cdots \text{``}=\text{''}\ a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ddots}}}}. \tag{7.25}$$

We emphasize that we have actually not proved that ξ is *equal* to the infinite continued fraction on the far right (hence, the quotation marks)! But, as a consequence of the following theorem, this equality follows; then the continued fraction in (7.25) is called the **canonical (simple) continued fraction expansion** of ξ.

THEOREM 7.14 (**Continued fraction convergence theorem**). *Let ξ_0, ξ_1, ξ_2, ... be any sequence of real numbers with $\xi_n > 0$ for $n \geq 1$ and suppose that these numbers are related by*

$$\xi_n = a_n + \frac{b_{n+1}}{\xi_{n+1}} \quad , \quad n = 0, 1, 2, \ldots,$$

for sequences of real numbers $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=1}^{\infty}$ with $a_n, b_n > 0$ for $n \geq 1$ and which satisfy $\sum_{n=1}^{\infty} \frac{a_n a_{n+1}}{b_{n+1}} = \infty$. Then ξ_0 is equal to the continued fraction

$$\xi_0 = a_0 + \frac{b_1}{a_1 +} \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \frac{b_4}{a_4 +} \frac{b_5}{a_5 +} \cdots.$$

In particular, for any real number ξ, the canonical continued fraction expansion (7.25) *converges to ξ.*

PROOF. By Theorem 7.12, the continued fraction $a_0 + \frac{b_1}{a_1 +} \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \cdots$ converges. Let $\{c_k = p_k/q_k\}$ denote the convergents of this infinite continued fraction and let $\varepsilon > 0$. Then by Theorem 7.12, there is an N such that

$$n > N \quad \Longrightarrow \quad |c_n - c_{n-1}| = \frac{b_1 b_2 \cdots b_n}{q_n\, q_{n-1}} < \varepsilon.$$

Fix $n > N$ and consider the *finite* continued fraction obtained as in (7.25) by writing out ξ_0 to the n-th term:

$$\xi_0 = a_0 + \frac{b_1}{a_1 +} \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \cdots \frac{b_{n-1}}{+\, a_{n-1} +} \frac{b_n}{\xi_n}.$$

Let $\{c'_k = p'_k/q'_k\}$ denote the convergents of this finite continued fraction. Then observe that $p_k = p'_k$ and $q_k = q'_k$ for $k \le n-1$ and $c'_n = \xi_0$. Therefore, by our fundamental recurrence relations, we have

$$|\xi_0 - c_{n-1}| = |c'_n - c'_{n-1}| \le \frac{b_1 b_2 \cdots b_n}{q'_n\, q'_{n-1}} = \frac{b_1 b_2 \cdots b_n}{q'_n\, q_{n-1}}.$$

By the Wallis-Euler relations, we have

$$q'_n = \xi_n q'_{n-1} + b_n q'_{n-2} = \left(a_n + \frac{b_{n+1}}{\xi_{n+1}}\right) q_{n-1} + b_n q_{n-2} > a_n q_{n-1} + b_n q_{n-2} = q_n.$$

Hence,

$$|\xi_0 - c_{n-1}| \le \frac{b_1 b_2 \cdots b_n}{q'_n\, q_{n-1}} < \frac{b_1 b_2 \cdots b_n}{q_n\, q_{n-1}} < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $\xi_0 = \lim c_{n-1} = \xi$. □

Example 7.14. Consider $\xi_0 = \sqrt{3} = 1.73205\ldots$. In this case, $a_0 := \lfloor \xi_0 \rfloor = 1$. Thus,

$$\xi_1 := \frac{1}{\xi_0 - a_0} = \frac{1}{\sqrt{3}-1} = \frac{1+\sqrt{3}}{2} = 1.36602\ldots \quad \Longrightarrow \quad a_1 := \lfloor \xi_1 \rfloor = 1.$$

Therefore,

$$\xi_2 := \frac{1}{\xi_1 - a_1} = \frac{1}{\frac{1+\sqrt{3}}{2} - 1} = 1 + \sqrt{3} = 2.73205\ldots \quad \Longrightarrow \quad a_2 := \lfloor \xi_2 \rfloor = 2.$$

Hence,

$$\xi_3 := \frac{1}{\xi_2 - a_2} = \frac{1}{\sqrt{3}-1} = \frac{1+\sqrt{3}}{2} = 1.36602\ldots \quad \Longrightarrow \quad a_3 := \lfloor \xi_3 \rfloor = 1.$$

Here we notice that $\xi_3 = \xi_1$ and $a_3 = a_1$. Therefore,

$$\xi_4 := \frac{1}{\xi_3 - a_3} = \frac{1}{\xi_1 - a_1} = \xi_2 = 1 + \sqrt{3} \quad \Longrightarrow \quad a_4 := \lfloor \xi_4 \rfloor = \lfloor \xi_2 \rfloor = 2.$$

At this point, we see that we will get the repeating pattern $1, 2, 1, 2, \ldots$, so we conclude that

$$\sqrt{3} = \langle 1; 1, 2, 1, 2, 1, 2, \ldots \rangle = \langle 1; \overline{1, 2} \rangle,$$

where we indicate that the $1, 2$ pattern repeats by putting a bar over them.

Example 7.15. Here is a neat example concerning the Fibonacci and Lucas numbers; for other fascinating topics on these numbers, see Knott's fun website [**93**]. Let us find the continued fraction expansion of the irrational number $\xi_0 = \Phi/\sqrt{5}$ where Φ is the golden ratio $\Phi = \frac{1+\sqrt{5}}{2}$:

$$\xi_0 = \frac{\Phi}{\sqrt{5}} = \frac{1+\sqrt{5}}{2\sqrt{5}} = 0.72360679\ldots \quad \Longrightarrow \quad a_0 := \lfloor \xi_0 \rfloor = 0.$$

Thus,

$$\xi_1 := \frac{1}{\xi_0 - a_0} = \frac{1}{\xi_0} = \frac{2\sqrt{5}}{1+\sqrt{5}} = 1.3819660\ldots \quad \Longrightarrow \quad a_1 := \lfloor \xi_1 \rfloor = 1.$$

Therefore,

$$\xi_2 := \frac{1}{\xi_1 - a_1} = \frac{1}{\frac{2\sqrt{5}}{1+\sqrt{5}} - 1} = \frac{1+\sqrt{5}}{\sqrt{5}-1} = 2.6180339\ldots \quad \Longrightarrow \quad a_2 := \lfloor \xi_2 \rfloor = 2.$$

Hence,

$$\xi_3 := \frac{1}{\xi_2 - a_2} = \frac{1}{\frac{1+\sqrt{5}}{\sqrt{5}-1} - 2} = \frac{\sqrt{5}-1}{3-\sqrt{5}} = 1.2360679\ldots \quad \Longrightarrow \quad a_3 := \lfloor \xi_3 \rfloor = 1.$$

Thus,

$$\xi_4 := \frac{1}{\xi_3 - a_3} = \frac{1}{\frac{\sqrt{5}-1}{3-\sqrt{5}} - 1} = \frac{3-\sqrt{5}}{2\sqrt{5}-4} = \frac{1+\sqrt{5}}{2} = 1.6180339\ldots;$$

that is, $\xi_4 = \Phi$, and so, $a_4 := \lfloor \xi_4 \rfloor = 1$. Let us do this one more time:

$$\xi_5 := \frac{1}{\xi_4 - a_4} = \frac{1}{\frac{1+\sqrt{5}}{2} - 1} = \frac{2}{\sqrt{5}-1} = \frac{1+\sqrt{5}}{2} = \Phi,$$

and so, $a_5 = a_4 = 1$. Continuing on this process, we will get $\xi_n = \Phi$ and $a_n = 1$ for the rest of the n's. In conclusion, we have

$$\frac{\Phi}{\sqrt{5}} = \langle 0; 1, 2, 1, 1, 1, 1, \ldots \rangle = \langle 0; 1, 2, \overline{1} \rangle.$$

The convergents of this continued fraction are fascinating. Recall that the Fibonacci sequence $\{F_n\}$, named after Leonardo Pisano Fibonacci (1170–1250), is defined as $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$, which gives the sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots.$$

The **Lucas numbers** $\{L_n\}$, named after François Lucas (1842–1891), are defined by

$$L_0 := 2\ ,\ L_1 = 1\ ,\quad L_n = L_{n-1} + L_{n-2}\ \ ,\ n = 2, 3, 4, \ldots,$$

and which give the sequence

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \ldots$$

If you work out the convergents of $\frac{\Phi}{\sqrt{5}} = \langle 0; 1, 2, 1, 1, 1, 1, \ldots \rangle$ what you get is the fascinating result:

$$\boxed{\begin{gathered}\frac{\Phi}{\sqrt{5}} = \langle 0; 1, 2, \overline{1} \rangle \ \text{ has convergents} \\ \frac{0}{2}, \frac{1}{1}, \frac{2}{3}, \frac{3}{4}, \frac{5}{7}, \frac{8}{11}, \frac{13}{18}, \frac{21}{29}, \frac{34}{47}, \frac{55}{76}, \frac{89}{123}, \ldots = \frac{\text{Fibonacci numbers}}{\text{Lucas numbers}};\end{gathered}} \tag{7.26}$$

of course, we do miss the other 1 in the Fibonacci sequence. For more fascinating facts on Fibonacci numbers see Problem 7. Finally, we remark that the canonical simple fraction expansion of a real number is unique, see Problem 8.

7.4.4. The numbers π and e. We now discuss the continued fraction expansions for the famous numbers π and e. Consider π first:

$$\xi_0 = \pi = 3.141592653\ldots \quad \Longrightarrow \quad a_0 := \lfloor \xi_0 \rfloor = 3.$$

Thus,

$$\xi_1 := \frac{1}{\pi - 3} = \frac{1}{0.141592653\ldots} = 7.062513305\ldots \quad \Longrightarrow \quad a_1 := \lfloor \xi_1 \rfloor = 7.$$

Therefore,

$$\xi_2 := \frac{1}{\xi_1 - a_1} = \frac{1}{0.062513305\ldots} = 15.99659440\ldots \quad \Longrightarrow \quad a_2 := \lfloor \xi_2 \rfloor = 15.$$

Hence,

$$\xi_3 := \frac{1}{\xi_2 - a_2} = \frac{1}{0.996594407\ldots} = 1.00341723\ldots \quad \Longrightarrow \quad a_3 := \lfloor \xi_3 \rfloor = 1.$$

Let us do this one more time:

$$\xi_4 := \frac{1}{\xi_3 - a_3} = \frac{1}{0.003417231\ldots} = 292.6345908\ldots \quad \Longrightarrow \quad a_4 := \lfloor \xi_4 \rfloor = 292.$$

Continuing this process (at Davis' Broadway cafe and after 314 free refills), we get

$$\pi = \langle 3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, \ldots \rangle. \tag{7.27}$$

Unfortunately (or perhaps fortunately) there is no known pattern that the partial quotients follow! The first few convergents for $\pi = 3.141592653\ldots$ are

$$c_0 = 3 \ , \ c_1 = \frac{22}{7} = 3.142857142\ldots \ , \ c_2 = \frac{333}{106} = 3.141509433\ldots$$

$$c_4 = \frac{355}{113} = 3.141592920\ldots \ , \ c_5 = \frac{103993}{33102} = 3.141592653\ldots$$

In stark contrast to π, Euler's number e has a shockingly simple pattern, which we ask you to work out in Problem 2:

$$e = \langle 2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots \rangle$$

We will prove that this pattern continues in Section 7.7!

7.4.5. Irrationality. We now discuss when continued fractions represent irrational numbers (cf. [**120**]).

THEOREM 7.15. *Let $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=1}^{\infty}$ be sequences rational numbers such that $a_n, b_n > 0$ for $n \geq 1$, $0 < b_n \leq a_n$ for all n sufficiently large, and $\sum_{n=1}^{\infty} \frac{a_n a_{n+1}}{b_{n+1}} = \infty$. Then the real number*

$$\xi = a_0 + \frac{b_1}{a_1 +} \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \frac{b_4}{a_4 +} \frac{b_5}{a_5 +} \cdots \quad \textit{is irrational.}$$

PROOF. First of all, the continued fraction defining ξ converges by Theorem 7.12. Suppose that $0 < b_n \leq a_n$ for all $n \geq m+1$ with $m > 0$. Observe that if we define

$$\eta = a_m + \frac{b_{m+1}}{a_{m+1} +} \frac{b_{m+2}}{a_{m+2} +} \frac{b_{m+3}}{a_{m+3} +} \cdots,$$

which also converges by Theorem 7.12, then $\eta > a_m > 0$ and we can write

$$\xi = a_0 + \frac{b_1}{a_1 +} \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \cdots + \frac{b_m}{\eta}.$$

By Theorem 7.4, we know that

$$\xi = a_0 + \frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\,\frac{b_3}{a_3+}\,\cdots\,\frac{b_m}{+\ \eta} = \frac{\eta p_m + b_m p_{m-1}}{\eta q_m + b_m q_{m-1}}.$$

Solving the last equation for η, we get

$$\xi = \frac{\eta p_m + b_m p_{m-1}}{\eta q_m + b_m q_{m-1}} \quad \Longleftrightarrow \quad \eta = \frac{\xi b_m q_{m-1} - b_m p_{m-1}}{p_m - \xi q_m}.$$

Note that since $\eta > a_m$, we have $\xi \neq p_m/q_m$. Since all the a_n, b_n's are rational, it follows that ξ is irrational if and only if η is irrational. Thus, all we have to do is prove that η is irrational. Since a_m is rational, all we have to do is prove that $\frac{b_{m+1}}{a_{m+1}+}\,\frac{b_{m+2}}{a_{m+2}+}\,\frac{b_{m+3}}{a_{m+3}+}\,\cdots$ is irrational, where $0 < b_n \leq a_n$ for all $n \geq m+1$. In conclusion, we might as well assume from the start that

$$\xi = \frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\,\frac{b_3}{a_3+}\,\frac{b_4}{a_4+}\,\frac{b_5}{a_5+}\,\cdots$$

where $0 < b_n \leq a_n$ for all n. We shall do this for the rest of the proof. Assume, by way of contradiction, that ξ is *rational*. Define $\xi_n := \frac{b_n}{a_n+}\,\frac{b_{n+1}}{a_{n+1}+}\,\frac{b_{n+2}}{a_{n+2}+}\,\cdots$. Then for each $n = 1, 2, \ldots$, we have

$$\text{(7.28)} \qquad \xi_n = \frac{b_n}{a_n + \xi_{n+1}} \quad \Longrightarrow \quad \xi_{n+1} = \frac{b_n}{\xi_n} - a_n.$$

By assumption, we have $0 < b_n \leq a_n$ for all n. It follows that $\xi_n > 0$ for all n and therefore

$$\xi_n = \frac{b_n}{a_n + \xi_{n+1}} < \frac{b_n}{a_n} \leq 1,$$

therefore $0 < \xi_n < 1$ for all n. Since $\xi_0 = \xi$, which is rational by assumption, by the second equality in (7.28) and induction it follows that ξ_n is rational for all n. Since $0 < \xi_n < 1$ for all n, we can therefore write $\xi_n = s_n/t_n$ where $0 < s_n < t_n$ for all n with s_n and t_n relatively prime integers. Now from the second equality in (7.28) we see that

$$\frac{s_{n+1}}{t_{n+1}} = \xi_{n+1} = \frac{b_n}{\xi_n} - a_n = \frac{b_n t_n}{s_n} - a_n = \frac{b_n t_n - a_n s_n}{s_n}.$$

Hence,

$$s_n\, s_{n+1} = (b_n t_n - a_n s_n) t_{n+1}.$$

Thus, $t_{n+1} | s_n\, s_{n+1}$. By assumption, s_{n+1} and t_{n+1} are relatively prime, so t_{n+1} must divide s_n. In particular, $t_{n+1} < s_n$. However, $s_n < t_n$ by assumption, so $t_{n+1} < t_n$. In summary, $\{t_n\}$ is a sequence of positive integers satisfying

$$t_1 > t_2 > t_3 > \cdots > t_n > t_{n+1} > \cdots > 0,$$

which of course is an absurdity because we would eventually reach zero! □

Example 7.16. (**Irrationality of e, Proof III**) Since we already know that (see (7.14))

$$e = 2 + \frac{2}{2+}\,\frac{3}{3+}\,\frac{4}{4+}\,\frac{5}{5+}\,\cdots,$$

we certainly have $b_n \leq a_n$ for all n, hence e is irrational!

As another application of this theorem, we get

Corollary 7.16. *Any infinite simple continued fraction represents an irrational number. In particular, a real number is irrational if and only if it can be represented by an infinite simple continued fraction.*

Indeed, for a simple continued fraction we have $b_n = 1$ for all n, so $0 < b_n \le a_n$ for all $n \ge 1$ holds.

Exercises 7.4.

1. (a) Use the simple continued fraction algorithm to the find the expansions of

$$(a)\ \sqrt{2}\quad,\quad (b)\ \frac{1-\sqrt{8}}{2}\quad,\quad (c)\ \sqrt{19}\quad,\quad (d)\ 3.14159\quad,\quad (e)\ \sqrt{7}.$$

   (b) Find the value of the continued fraction expansions

$$(a)\ 4+\frac{2}{8+}\,\frac{2}{8+}\,\frac{2}{8+}\cdots\quad,\quad (b)\ \langle\overline{3}\rangle = \langle 3;3,3,3,3,3,\ldots\rangle.$$

   The continued fraction in (a) was studied by Pietro Antonio Cataldi (1548–1626) and is one of the earliest infinite continued fractions on record.
2. In Section 7.7, we will prove the conjectures you make in (a) and (b) below.
   (a) Using a calculator, we find that $e \approx 2.718281828$. Verify that $2.718281828 = \langle 2,1,2,1,1,4,1,1,6,\ldots\rangle$. From this, conjecture a formula for a_n, $n = 0,1,2,3,\ldots$, in the canonical continued fraction expansion for e.
   (b) Using a calculator, we find that $\frac{e+1}{e-1} \approx 2.1639534137$. Find a_0, a_1, a_2, a_3 in the canonical continued fraction expansion for 2.1639534137 and conjecture a formula for a_n, $n = 0,1,2,3,\ldots$, in the canonical continued fraction expansion for $\frac{e+1}{e-1}$.
3. Let $n \in \mathbb{N}$. Prove that $\sqrt{n^2+1} = \langle n; \overline{2n}\rangle$ by using the simple continued fraction algorithm on $\sqrt{n^2+1}$. Using the same technique, find the canonical expansion of $\sqrt{n^2+2}$. (See Problem 5 below for other proofs.)
4. In this problem we show that any positive real number can be written as two different infinite continued fractions. Let a be a positive real number. Prove that

$$a = 1 + \cfrac{k}{1+\cfrac{k}{1+\cfrac{k}{1+\ddots}}} = \cfrac{\ell}{1+\cfrac{\ell}{1+\cfrac{\ell}{1+\ddots}}},$$

   where $k = a^2 - a$ and $\ell = a^2 + a$. Suggestion: Link the limits of continued fractions on the right to the quadratic equations $x^2 - x - k = 0$ and $x^2 + x - \ell = 0$, respectively. Find neat infinite continued fractions for $1, 2$, and 3.
5. Let x be any positive real number and suppose that $x^2 - ax - b = 0$ where a, b are positive. Prove that

$$x = a + \frac{b}{a+}\,\frac{b}{a+}\,\frac{b}{a+}\,\frac{b}{a+}\,\frac{b}{a+}\cdots.$$

   Using this, prove that

$$\sqrt{\alpha^2+\beta} = \alpha + \frac{\beta}{2\alpha+}\,\frac{\beta}{2\alpha+}\,\frac{\beta}{2\alpha+}\,\frac{\beta}{2\alpha+}\cdots.$$

6. (a) Prove that a continued fraction $a_0 + \frac{b_1}{a_1+}\frac{b_2}{a_2+}\frac{b_3}{a_3+}\cdots$ converges if and only if

$$c_0 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} b_1 b_2 \cdots b_n}{q_n\, q_{n-1}}$$

   converges, in which case, this sum is exactly $a_0 + \frac{b_1}{a_1+}\frac{b_2}{a_2+}\frac{b_3}{a_3+}\cdots$. Suggestion: Consider the telescoping sum $c_n = c_0 + (c_1 - c_0) + (c_2 - c_1) + \cdots + (c_n - c_{n-1})$. In particular, for a simple continued fraction $\xi = \langle a_0; a_1, a_2, a_3, \ldots \rangle$, we have

$$\xi = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{q_n\, q_{n-1}}.$$

(b) Assume that $\xi = a_0 + \frac{b_1}{a_1 +} \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \cdots$ converges. Prove that

$$\xi = c_0 + \sum_{n=2}^{\infty} \frac{(-1)^n a_n b_1 b_2 \cdots b_{n-1}}{q_n\, q_{n-2}}.$$

In particular, for a simple continued fraction $\xi = \langle a_0; a_1, a_2, a_3, \ldots \rangle$, we have

$$\xi = 1 + \sum_{n=2}^{\infty} \frac{(-1)^n a_n}{q_n\, q_{n-2}}.$$

7. Let $\{c_n\}$ be the convergents of $\Phi = \langle 1; 1, 1, 1, 1, 1, 1, \ldots \rangle$.
   (1) Prove that for $n \geq 1$, we have $\frac{F_{n+1}}{F_n} = c_{n-1}$. (That is, $p_n = F_{n+2}$ and $q_n = F_{n+1}$.) Conclude that

$$\boxed{\Phi = \lim_{n \to \infty} \frac{F_{n+1}}{F_n}\ ,}$$

   a beautiful (but nontrivial) fact!
   (2) Using the previous problem, prove the incredibly beautiful formulas

$$\boxed{\Phi = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1}} \qquad \text{and} \qquad \Phi^{-1} = \sum_{n=2}^{\infty} \frac{(-1)^n}{F_n F_{n+2}}.}$$

8. Let $\alpha = \langle a_0; a_1, a_2, \ldots \rangle$, $\beta = \langle b_0; b_1, b_2, \ldots \rangle$ be infinite simple continued fractions. Prove that if $\alpha = \beta$, then $a_k = b_k$ for all $k = 0, 1, 2, \ldots$, which shows that the canonical simple fraction expansion of an irrational real number is unique. See Problem 6 in Exercises 7.3 for the rational case.
9. A continued fraction $a_0 + \frac{1}{a_1 +} \frac{1}{a_2 +} \frac{1}{a_3 +} \frac{1}{a_4 +} \cdots$ where the a_n are real numbers with $a_n > 0$ for $n \geq 1$ is said to be **unary**. In this problem we prove that a unary continued fraction converges if and only if $\sum a_n = \infty$. Henceforth, let $a_0 + \frac{1}{a_1 +} \frac{1}{a_2 +} \frac{1}{a_3 +} \cdots$ be unary.
   (i) Prove that $q_n \leq \prod_{k=1}^{n} (1 + a_k)$.
   (ii) Using the inequality derived in (9i), prove that if the unary continued fraction converges, then $\sum a_n = \infty$.
   (iii) Prove that

$$q_{2n} \geq 1 + a_1(a_2 + a_4 + \cdots + a_{2n}) \quad , \quad q_{2n-1} \geq a_1 + a_3 + \cdots + a_{2n-1},$$

   where the first inequality holds for $n \geq 1$ and the second for $n \geq 2$.
   (iv) Using the inequalities derived in (9iii), prove that if $\sum a_n = \infty$, then the unary continued fraction converges.

7.5. Diophantine approximations and the mystery of π solved!

For practical purposes, it is necessary to approximate irrational numbers by rational numbers. Also, if a rational number has a very large denominator, e.g. $\frac{1234567}{1211109876543 21}$, then it is hard to work with, so for practical purposes it would be nice to have a "good" approximation to such a rational number by a rational number with a more manageable denominator. Diophantine approximations is the subject of finding "good" or even "best" rational approximations to real numbers. Continued fractions turn out to play a very important role in this subject, to which this section is devoted. We start with a journey concerning the mysterious fraction representations of π.

7.5.1. The mystery of π and good and best approximations. Here we review some approximations to $\pi = 3.14159265\ldots$ that have been discovered throughout the centuries (see Section 4.10 for a thorough study):

(1) 3 in the Holy Bible circa 1000 B.C. by the Hebrews; See Book of I Kings, Chapter 7, verse 23, and Book of II Chronicles, Chapter 4, verse 2:

> And he made a molten sea, ten cubits from the one brim to the other: it was round all about, and his height was five cubits: and a line of thirty cubits did compass it about. I Kings 7:23.

(2) $22/7 = 3.14285714\ldots$ (correct to two decimal places) by Archimedes of Syracuse (287–212) circa 250 B.C.

(3) $333/106 = 3.14150943\ldots$ (correct to four decimal places), a lower bound found by Adriaan Anthoniszoon (1527–1607) circa 1600 A.D.

(4) $355/113 = 3.14159292\ldots$ (correct to six decimal places) by Tsu Chung-Chi (429–501) circa 500 A.D.

Hmmm... these numbers certainly seem familiar! These numbers are exactly the first four convergents of the continued fraction expansion of π that we worked out in Subsection 7.4.4! From this example, it seems like approximating real numbers by rational numbers is intimately related to continued fractions; this is indeed the case as we shall see. To start our adventure in approximations, we start with the concepts of "good" and "best" approximations.

A rational number p/q is called a **good approximation** to a real number ξ if[2]

$$\boxed{\text{for all rational } \frac{a}{b} \neq \frac{p}{q} \text{ with } 1 \le b \le q, \text{ we have } \left|\xi - \frac{p}{q}\right| < \left|\xi - \frac{a}{b}\right|;}$$

in other words, we cannot get closer to the real number ξ with a different rational number having a denominator $\le q$.

Example 7.17. $4/1$ is *not* a good approximation to π because $3/1$, which has an equal denominator, is closer to π:

$$\left|\pi - \frac{3}{1}\right| = 0.141592\ldots < \left|\pi - \frac{4}{1}\right| = 0.858407\ldots.$$

Example 7.18. As another example, $7/2$ is *not* a good approximation to π because $3/1$, which has a smaller denominator than $7/2$, is closer to π:

$$\left|\pi - \frac{3}{1}\right| = 0.141592\ldots < \left|\pi - \frac{7}{2}\right| = 0.358407\ldots.$$

This example shows that you wouldn't want to approximate π with $7/2$ because you can approximate it with the "simpler" number $3/1$ that has a smaller denominator.

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[2]Warning: Some authors define good approximation as: $\frac{p}{q}$ is a good approximation to ξ if for all rational $\frac{a}{b}$ with $1 \le b < q$, we have $\left|\xi - \frac{p}{q}\right| < \left|\xi - \frac{a}{b}\right|$. This definition, although only slightly different from ours, makes some proofs *considerably easier*. Moreover, with this definition, $1,000,000/1$ is a good approximation to π (why?)! (In fact, *any* integer, no matter how big, is a good approximation to π.) On the other hand, with our definition, the only integer that is a good approximation to π is 3. This is why we like our definition. Also, some authors define best approximation as: $\frac{p}{q}$ is a best approximation to ξ if for all rational $\frac{a}{b}$ with $1 \le b < q$, we have $|q\xi - p| < |b\xi - a|$; with this definition of "best," one can shorten the proof of Theorem 7.20 — but then one must live with the fact that $1,000,000/1$ is a best approximation to π.

Example 7.19. On the other hand, 13/4 is a good approximation to π. This is because

$$\left|\pi - \frac{13}{4}\right| = 0.108407\ldots,$$

and there are no fractions closer to π with denominator 4, and the closest distinct fractions with the smaller denominators 1, 2, and 3 are 3/1, 7/2, and 10/3, which satisfy

$$\left|\pi - \frac{3}{1}\right| = 0.141592\ldots \quad , \quad \left|\pi - \frac{7}{2}\right| = 0.358407\ldots \quad , \quad \left|\pi - \frac{10}{3}\right| = 0.191740\ldots.$$

Thus,

$$\text{for all rational } \frac{a}{b} \neq \frac{13}{4} \text{ with } 1 \leq b \leq 4, \text{ we have } \left|\pi - \frac{13}{4}\right| < \left|\pi - \frac{a}{b}\right|.$$

Now one can argue: Is 13/4 really that great of an approximation to π? For although 3/1 is not as close to π, it is certainly much easier to work with than 13/4 because of the larger denominator 4 — moreover, we have $13/4 = 3.25$, so we didn't even gain a single decimal place of accuracy in going from 3.00 to 3.25! These are definitely valid arguments. One can also see the validity of this argument by combining fractions in the inequality in the definition of good approximation: p/q is a good approximation to ξ if

$$\text{for all rational } \frac{a}{b} \neq \frac{p}{q} \text{ with } 1 \leq b \leq q, \text{ we have } \frac{|q\xi - p|}{q} < \frac{|b\xi - a|}{b},$$

where we used that $q, b > 0$. Here, we can see that $\frac{|q\xi-p|}{q} < \frac{|b\xi-a|}{b}$ may hold not because p/q is dramatically much closer to ξ than is a/b but simply because q is a lot larger than b (like in the case 13/4 and 3/1 where 4 is much larger than 1). To try and correct this somewhat misleading notion of "good" we introduce the concept of a "best" approximation by clearing the denominators.

A rational number p/q is called a **best approximation** to a real number ξ if

$$\boxed{\text{for all rational } \frac{a}{b} \neq \frac{p}{q} \text{ with } 1 \leq b \leq q, \text{ we have } \left|q\xi - p\right| < \left|b\xi - a\right|.}$$

Example 7.20. We can see that $p/q = 13/4$ is *not* a best approximation to π because with $a/b = 3/1$, we have $1 \leq 1 \leq 4$ yet

$$\left|4 \cdot \pi - 13\right| = 0.433629\ldots \not< \left|1 \cdot \pi - 3\right| = 0.141592\ldots.$$

Thus, 13/4 is a good approximation to π but is far from a best approximation.

In the following proposition, we show that any best approximation is a good one.

PROPOSITION 7.17. *A best approximation is a good one, but not vice versa.*

PROOF. We already gave an example showing that a good approximation may not be a best one, so let p/q be a best approximation to ξ; we shall prove that p/q is a good one too. Let $a/b \neq p/q$ be rational with $1 \leq b \leq q$. Then $|q\xi - p| < |b - \xi a|$ since p/q is a best approximation, and also, $\frac{1}{q} \leq \frac{1}{b}$ since $b \leq q$, hence

$$\left|\xi - \frac{p}{q}\right| = \frac{|q\xi - p|}{q} < \frac{|b\xi - a|}{q} \leq \frac{|b\xi - a|}{b} = \left|\xi - \frac{a}{b}\right| \quad \Longrightarrow \quad \left|\xi - \frac{p}{q}\right| < \left|\xi - \frac{a}{b}\right|.$$

This shows that p/q is a good approximation. $\square$

In the following subsection, we shall prove the best approximation theorem, Theorem 7.20, which states that

> ***(Best approximation theorem)*** *Every best approximation of a real number (rational or irrational) is a convergent of its canonical continued fraction expansion and conversely, each of the convergents $c_1, c_2, c_3, \ldots$ is a best approximation.*

Unfortunately, the proof of this theorem is probably one of the hardest/longest ones we've had the pleasure of meeting so far in our journey through this book; I don't know how to make the proof significantly easier without changing the definition of "best" as described in a footnote a couple pages back.[3] We suggest that you skip the proofs of Lemma 7.19 and Theorem 7.20 at a first reading; the proof of Theorem 7.18 is not bad and the readings in between the proofs are illustrative.

7.5.2. Approximations, convergents, and the "most irrational" of all irrational numbers. The objective of this subsection is to understand how convergents approximate real numbers. In the following theorem, we show that the convergents of the simple continued fraction of a real number ξ get increasingly closer to ξ. (See Problem 4 for the general case of nonsimple continued fractions.)

THEOREM 7.18 (**Fundamental approximation theorem**). *Let ξ be an irrational number and let $\{c_n = p_n/q_n\}$ be the convergents of its canonical continued fraction. Then the following inequalities hold:*

$$\left|\xi - c_n\right| < \frac{1}{q_n q_{n+1}}, \quad \left|\xi - c_{n+1}\right| < \left|\xi - c_n\right|, \quad \left|q_{n+1}\xi - p_{n+1}\right| < \left|q_n\xi - p_n\right|.$$

If ξ is a rational number and the convergent c_{n+1} is defined (that is, if $\xi \neq c_n$), then these inequalities still hold.

PROOF. We prove this theorem for ξ irrational; the rational case is proved using a similar argument, which we leave to you if you're interested. The proof of this theorem is very simple. We just need the inequalities

$$c_n < c_{n+2} < \xi < c_{n+1} \quad \text{or} \quad c_{n+1} < \xi < c_{n+2} < c_n, \tag{7.29}$$

depending on whether n is even or odd, respectively, and the fundamental recurrence relations (see Corollary 7.6):

$$c_{n+1} - c_n = \frac{(-1)^n}{q_n\, q_{n+1}} \quad , \quad c_{n+2} - c_n = \frac{(-1)^n a_{n+2}}{q_n\, q_{n+2}}. \tag{7.30}$$

Now the first inequality of our theorem follows easily:

$$\left|\xi - c_n\right| \overset{\text{by (7.29)}}{<} \left|c_{n+1} - c_n\right| \overset{\text{by (7.30)}}{=} \left|\frac{(-1)^n}{q_n\, q_{n+1}}\right| = \frac{1}{q_n\, q_{n+1}}.$$

[3]*Unfortunately what is little recognized is that the most worthwhile scientific books are those in which the author clearly indicates what he does not know; for an author most hurts his readers by concealing difficulties. Evariste Galois (1811–1832).* [**142**].

We now prove that $|q_{n+1}\xi - p_{n+1}| < |q_n\xi - p_n|$. To prove this, we work on the left and right-hand sides separately. For the left-hand side, we have

$$\begin{aligned}|q_{n+1}\xi - p_{n+1}| = q_{n+1}\left|\xi - \frac{p_{n+1}}{q_{n+1}}\right| = q_{n+1}|\xi - c_{n+1}| &< q_{n+1}|c_{n+2} - c_{n+1}| \text{ by (7.29)}\\ &= q_{n+1}\frac{1}{q_{n+1}\, q_{n+2}} \quad \text{by (7.30)}\\ &= \frac{1}{q_{n+2}}.\end{aligned}$$

Hence, $\frac{1}{q_{n+2}} > |q_{n+1}\xi - p_{n+1}|$. Now,

$$\begin{aligned}|q_n\xi - p_n| = q_n\left|\xi - \frac{p_n}{q_n}\right| = q_n|\xi - c_n| &> q_n|c_{n+2} - c_n| \qquad \text{by (7.29)}\\ &= q_n\frac{a_{n+2}}{q_n\, q_{n+2}} \qquad \text{by (7.30)}\\ &= \frac{a_{n+2}}{q_{n+2}} \geq \frac{1}{q_{n+2}} > |q_{n+1}\xi - p_{n+1}|.\end{aligned}$$

This proves our third inequality. Finally, using what we just proved, and that

$$q_{n+1} = a_{n+1}q_n + q_{n-1} \geq q_n + q_{n-1} > q_n \quad \Longrightarrow \quad \frac{1}{q_{n+1}} < \frac{1}{q_n},$$

we see that

$$\begin{aligned}|\xi - c_{n+1}| = \left|\xi - \frac{p_{n+1}}{q_{n+1}}\right| &= \frac{1}{q_{n+1}}|q_{n+1}\xi - p_{n+1}|\\ &< \frac{1}{q_{n+1}}|q_n\xi - p_n|\\ &< \frac{1}{q_n}|q_n\xi - p_n| = \left|\xi - \frac{p_n}{q_n}\right| = |\xi - c_n|.\end{aligned}$$

□

It is important to only use the *canonical* expansion when ξ is rational. This is because the statement that $|q_{n+1}\xi - p_{n+1}| < |q_n\xi - p_n|$ may not *not* be true if we don't use the canonical expansion.

Example 7.21. Consider 5/3, which has the canonical expansion:

$$\frac{5}{3} = \langle 1; 1, 2\rangle = 1 + \cfrac{1}{1 + \cfrac{1}{2}}.$$

We can write this as the noncanonical expansion by breaking up the 2:

$$\xi = \langle 1; 1, 1, 1\rangle = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}} = \frac{5}{3}.$$

The convergents for ξ are $c_0 = 1/1$, $c_2 = 2/1$, $c_3 = 3/2$, and $\xi = c_4 = 5/3$. In this case,

$$|q_3\xi - p_3| = \left|2 \cdot \frac{5}{3} - 3\right| = \frac{1}{3} = \left|1 \cdot \frac{5}{3} - 2\right| = |q_2\xi - p_2|,$$

so for this example, $|q_2\xi - p_2| \not< |q_2\xi - p_2|$.

We now discuss the "most irrational" of all irrational numbers. From the best approximation theorem (Theorem 7.20 we'll prove in a moment) we know that the best approximations of a real number ξ are convergents and from the fundamental approximation theorem 7.18, we have the error estimate

$$\big|\xi - c_n\big| < \frac{1}{q_n q_{n+1}} \quad \Longrightarrow \quad \big|q_n \xi - p_n\big| < \frac{1}{q_{n+1}}. \tag{7.31}$$

This shows you that the larger the q_n's are, the better the best approximations are. Since the q_n's are determined by the recurrence relation $q_n = a_n q_{n-1} + q_{n-2}$, we see that the larger the a_n's are, the larger the q_n's are. In summary, ξ can be approximated very "good" by rational numbers when it has *large* a_n's and very "bad" by rational numbers when it has *small* a_n's.

Example 7.22. Here is a "good" example: Recall from (7.27) the continued fraction for π: $\pi = \langle 3; 7, 15, 1, 292, 1, 1, 1, 2, 1, \ldots\rangle$, which has convergents $c_0 = 3$, $c_1 = \frac{22}{7}$, $c_2 = \frac{333}{106}$, $c_3 = \frac{355}{113}$, $c_4 = \frac{103993}{33102}$, ... Because of the large number $a_4 = 292$, we see from (7.31) that we can approximate π very nicely with c_3: Using the left-hand equation in (7.31), we see that

$$\big|\pi - c_3\big| < \frac{1}{q_3 q_4} = \frac{1}{113 \cdot 33102} = 0.000000267\ldots,$$

which implies that $c_3 = \frac{355}{113}$ approximates π to within six decimal places! (Just to check, note that $\pi = 3.14159265\ldots$ and $\frac{355}{113} = 3.14159292\ldots$.) It's amazing how many decimal places of accuracy we can get with just taking the c_3 convergent!

Example 7.23. (**The "most irrational" number**) Here is a "bad" example: From our discussion after (7.31), we saw that the smaller the a_n's are, the worse it can be approximated by rationals. Of course, since 1 is the smallest natural number, we can consider the golden ratio

$$\Phi = \frac{1 + \sqrt{5}}{2} = \langle 1; 1, 1, 1, 1, 1, 1, 1, \ldots\rangle = 1.6180339887\ldots$$

as being the "worst" of all irrational numbers that can be approximated by rational numbers. Indeed, we saw that we could get six decimal places of π by just taking c_3; for Φ we need c_{18}! (Just to check, we find that $c_{17} = \frac{4181}{2584} = 1.6180340557\ldots$ — not quite six decimals — and $c_{18} = \frac{6765}{4181} = 1.618033963\ldots$ — got the sixth one. Also notice the large denominator 4181 just to get six decimals.) Therefore, Φ wins the prize for the "most irrational" number in that it's the "farthest" from the rationals! We continue our discussion on "most irrational" in Subsection 7.10.3.

We now show that best approximations are exactly convergents; this is one of the most important properties of continued fractions. In the following lemma we begin by showing that best approximations are convergents and in the next theorem we prove the converse.

Lemma 7.19. *Every best approximation of a real number (rational or irrational) is a convergent of the simple continued fraction expansion of the real number.*

Proof. Let p/q be a best approximation of a real number ξ. Just so that we don't have to think about terminating continued fractions (the rational numbers) let's assume that ξ is irrational; the case when ξ is rational is handled in a similar manner. We need to show that p/q equals one of these convergents

$p_0/q_0, p_1/q_1, p_2/q_2, \ldots$ of the continued fraction expansion of $\xi = \langle a_0; a_1, a_2, \ldots \rangle$. For sake of contradiction, assume that p/q is not one of the convergents; we shall derive a contradiction. Recall that the convergents of ξ satisfy

$$\frac{a_0}{1} = \frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \cdots < \xi < \cdots < \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1}. \tag{7.32}$$

Step 1: We first prove that p/q must lie between the two extremities of this list:

$$\frac{a_0}{1} = \frac{p_0}{q_0} < \frac{p}{q} < \frac{p_1}{q_1}$$

Suppose this is not true. Then $p/q < p_0/q_0$ or $p_1/q_1 < p/q$ (there cannot be equalities here because p/q is by assumption not equal to a convergent). Suppose that $p/q < p_0/q_0$. Then from (7.32), we see that

$$\frac{p}{q} < a_0 < \xi.$$

This shows that $|\xi - a_0| < |\xi - p/q|$, which implies that

$$|1 \cdot \xi - a_0| \leq q \cdot |\xi - a_0| < q \cdot |\xi - p/q| = |q\xi - p|.$$

Since $1 \leq 1 \leq q$, this contradicts that p/q is a best approximation to ξ, therefore $p/q < p_0/q_0$ cannot hold. On the other hand, suppose that $p_1/q_1 < p/q$. Then from (7.32), we see that

$$\xi < \frac{p_1}{q_1} < \frac{p}{q}.$$

This shows that $q\xi < qp_1/q_1 < p$, and subtracting p from everything, we get

$$q\xi - p < q\frac{p_1}{q_1} - p < 0,$$

which implies that

$$0 < \frac{|qp_1 - pq_1|}{q_1} = \left|\frac{qp_1}{q_1} - p\right| < \left|q\xi - p\right|.$$

In particular, $|qp_1 - pq_1|$, which is an integer, is not zero, so $|qp_1 - pq_1| \geq 1$. Thus,

$$\frac{1}{q_1} < \left|q\xi - p\right|.$$

Now recall that $q_1 = a_1$ and $a_1 := \lfloor \xi_1 \rfloor < \xi_1$ where we write $\xi = a_0 + \frac{1}{\xi_1}$, so

$$|\xi - \frac{a_0}{1}| = \frac{1}{\xi_1} < \frac{1}{q_1} < \left|q\xi - p\right|.$$

This contradicts that p/q is a best approximation to ξ and completes the proof of **Step 1**. We now move to

Step 2: We now complete the proof. By **Step 1**, we have $p_0/q_0 < p/q < p_1/q_1$, so by the inequalities (7.32), we see that

$$\frac{p_n}{q_n} < \frac{p}{q} < \frac{p_{n+2}}{q_{n+2}} < \xi < \frac{p_{n+1}}{q_{n+1}} \quad \text{or} \quad \frac{p_{n+1}}{q_{n+1}} < \xi < \frac{p_{n+2}}{q_{n+2}} < \frac{p}{q} < \frac{p_n}{q_n}, \tag{7.33}$$

depending on whether p/q is to the left or right of ξ. In either case, the inequalities (7.33) show that

$$\frac{|pq_n - p_nq|}{qq_n} = \left|\frac{p}{q} - \frac{p_n}{q_n}\right| < \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| = \left|c_{n+1} - c_n\right| = \frac{1}{q_nq_{n+1}},$$

where we used the fundamental recurrence relation at the end. Since $p/q \neq p_n/q_n$, the integer $|pq_n - p_n q|$ must be ≥ 1, so we actually have

$$\frac{1}{qq_n} < \frac{1}{q_n q_{n+1}} \implies q_{n+1} < q. \tag{7.34}$$

The inequalities (7.33) also imply that

$$\frac{|pq_{n+2} - p_{n+2}q|}{qq_{n+2}} = \left|\frac{p}{q} - \frac{p_n}{q_n}\right| < \left|\xi - \frac{p}{q}\right|.$$

Since $p/q \neq p_{n+2}/q_{n+2}$, the integer $|pq_{n+2} - p_{n+2}q|$ must be ≥ 1, so we actually have

$$\frac{1}{qq_{n+2}} < \left|\xi - \frac{p}{q}\right| \implies \frac{1}{q_{n+2}} < \left|q\xi - p\right|.$$

Also, by the fundamental approximation theorem, Theorem 7.18, we have

$$|q_{n+1}\xi - p_{n+1}| = q_{n+1}\left|\xi - \frac{p_{n+1}}{q_{n+1}}\right| \leq q_{n+1} \cdot \frac{1}{q_{n+1}q_{n+2}} = \frac{1}{q_{n+2}},$$

therefore

$$|q_{n+1}\xi - p_{n+1}| < \left|q\xi - p\right|.$$

However, this inequality plus the fact that $q_{n+1} < q$ from (7.34) shows that p/q is not a best approximation to ξ. This contradiction completes **Step 2** and finishes our proof. ☐

We now prove the converse.

THEOREM 7.20 (**Best approximation theorem**). *Every best approximation of a real number (rational or irrational) is a convergent of its canonical continued fraction expansion and conversely, each of the convergents $c_1, c_2, c_3, \ldots$ is a best approximation.*

PROOF. We already showed that every best approximation is a convergent, so we just need to show that each convergent $c_1, c_2, c_3, \ldots$ is a best approximation. Let ξ be a real number with convergents $\{c_n = p_n/q_n\}$.

We prove that p_n/q_n is a best approximation using induction. We start with the $n = 1$ case. (This is the "easy" part of the proof.) We prove that p_1/q_1 is a best approximation by contradiction. If p_1/q_1 is not a best approximation, then by definition of best approximation, there must exist another fraction $a/b \neq p_1/q_1$ with $1 \leq b \leq q_1$ and

$$|b\xi - a| \leq |q_1\xi - p_1|.$$

Using this inequality we'll derive a contradiction. Note that this inequality implies, in particular, that in the case that ξ happens to be rational, $\xi_1 \neq p_1/q_1$. We derive a contradiction by simply working with the definition of the canonical expansion:

$$\xi = a_0 + \frac{1}{\xi_1}, \quad \text{where } \xi_1 > 1 \text{ and } a_1 := \lfloor \xi_1 \rfloor$$

and the definitions of q_1 and p_1:

$$q_1 = a_1 = \lfloor \xi_1 \rfloor \quad \text{and} \quad p_1 = a_0 a_1 + 1 = a_0 q_1 + 1. \tag{7.35}$$

Therefore, the inequality $|b\xi - a| \leq |q_1\xi - p_1|$ can be written as

$$\left|b\Big(a_0 + \frac{1}{\xi_1}\Big) - a\right| \leq \left|q_1\Big(a_0 + \frac{1}{\xi_1}\Big) - \big(a_0 q_1 + 1\big)\right|.$$

After rearrangements and cancellations, this takes the form

$$\left|ba_0 + \frac{b}{\xi_1} - a\right| \le \left|\frac{q_1}{\xi_1} - 1\right| \quad \Longrightarrow \quad \left|a - ba_0 - \frac{b}{\xi_1}\right| \le \left|1 - \frac{q_1}{\xi_1}\right|. \tag{7.36}$$

Since (see (7.35)) $0 < q_1 = \lfloor \xi_1 \rfloor \le \xi_1$ it follows that $0 < q_1/\xi_1 \le 1$, so $|1 - q_1/\xi_1| < 1$. Hence, by (7.36), we have

$$\left|a - ba_0 - \frac{b}{\xi_1}\right| < 1 \quad \Longrightarrow \quad \frac{b}{\xi_1} - 1 < a - ba_0 < 1 + \frac{b}{\xi_1}.$$

By assumption, $1 \le b \le q_1 = \lfloor \xi_1 \rfloor$, so $0 < b/\xi_1 \le 1$, and hence $-1 < a - ba_0 < 2$. Thus, as $a - ba_0$ is an integer, this integer must be either 0 or 1. If $a - ba_0 = 0$, then from (7.36), we have

$$\left|\frac{b}{\xi_1}\right| \le \left|1 - \frac{q_1}{\xi_1}\right| \quad \Longrightarrow \quad b \le \left|\xi_1 - q_1\right| = \xi_1 - \lfloor \xi_1 \rfloor < 1,$$

which shows that $b < 1$, an impossibility because $1 \le b$. On the other hand, if $a - ba_0 = 1$, then again by (7.36), we have

$$\left|1 - \frac{b}{\xi_1}\right| \le \left|1 - \frac{q_1}{\xi_1}\right| \quad \Longrightarrow \quad \left|\xi_1 - b\right| \le \left|\xi_1 - q_1\right| \quad \Longrightarrow \quad \xi_1 - b \le \xi_1 - q_1,$$

where we used that $\xi_1 - q_1 = \xi_1 - \lfloor \xi_1 \rfloor \ge 0$ and $b \le q_1$ so that $\xi_1 - b \ge 0$ as well. Cancelling off the ξ_1's, we see that $q_1 \le b$. Since $b \le q_1$, we must therefore have $b = q_1$. Then the equality $a - ba_0 = 1$ shows that (see (7.35))

$$a = a_0 b + 1 = a_0 q_1 + 1 = p_1.$$

Thus, $a/b = p_1/q_1$, another contradiction since we assumed from the start that $a/b \ne p_1/q_1$. We have thus finished proving the base case.

Assume now that p_n/q_n with $n \ge 1$ is a best approximation; we shall prove that p_{n+1}/q_{n+1} is a best approximation. (This is the "harder" part of this proof.) We may assume that $\xi \ne p_{n+1}/q_{n+1}$, for otherwise p_{n+1}/q_{n+1} is automatically a best approximation. From Theorem 7.18 we know that

$$|y\xi - x| < |q_n \xi - p_n| \qquad \text{for } x = p_{n+1} \quad \text{and} \quad y = q_{n+1}. \tag{7.37}$$

The idea for this part of the proof is simple but very time consuming: We minimize the left-hand side of this inequality over all rational x/y with $1 \le y \le q_{n+1}$, then prove that the minimizing x/y is a best approximation, and finally, prove that $x/y = p_{n+1}/q_{n+1}$. To minimize the left-hand side of (7.37), we first let q be the smallest denominator y of all rational numbers x/y with $y > 0$ such that

$$|y\xi - x| < |q_n \xi - p_n|. \tag{7.38}$$

(Such a q exists by well-ordering. To see this, just let $A \subseteq \mathbb{N}$ consist of all denominators $y > 0$ of rational numbers x/y such that (7.38) holds. By (7.37) we know that $q_{n+1} \in A$ so $A \ne \varnothing$. By well-ordering, the set A has a smallest element; this element is q.) Since q is the smallest denominator y satisfying (7.38) for all x/y with $y > 0$, by (7.37) we must have $q \le q_{n+1}$. Now let p be the natural number x that makes $|q\xi - x|$ the smallest. In particular,

$$|q\xi - p| < |q_n \xi - p_n|.$$

Note that p/q is reduced, for if $p = jx$ and $q = jy$ with $j \ge 2$ for some x, y, then $y < q$ and $|y\xi - x| \le j|y\xi - x| = |q\xi - p| < |q_n \xi - p_n|$, but this contradicts the definition of q as the smallest positive y satisfying (7.38). Also note that since

$|q\xi - p| < |q_n\xi - p_n|$ and p_n/q_n is a best approximation by hypothesis, we must have $q_n < q$. In summary, in addition to the definitions of q and p, we also have $q_n < q \le q_{n+1}$ and p/q is reduced.

We claim that p/q is a best approximation to ξ. Let us assume, just for a moment, that we have proved this. Then from Lemma 7.19 we know that p/q must be a convergent. In this case, $p/q = p_k/q_k$ for some k. Since p/q is reduced and so is p_k/q_k we must have $p = p_k$ and $q = q_k$. On the other hand, we know that $q_n < q \le q_{n+1}$, so k must be $n+1$. This shows that $p/q = p_{n+1}/q_{n+1}$ and our proof is finished, once we show that p/q is a best approximation.

For sake of contradiction, assume that p/q is not a best approximation. Then there must exist a rational $a/b \neq p/q$ such that $1 \le b \le q$ and

$$\big|b\xi - a\big| \le \big|q\xi - p\big|. \tag{7.39}$$

Since $|q\xi - p| < |q_n\xi - p_n|$ it follows that a/b satisfies $|b\xi - a| < |q_n\xi - p_n|$. Now q is, by definition, the smallest positive denominator y in a rational x/y satisfying $|y\xi - x| < |q_n\xi - p_n|$, so we must have $q \le b$. However, $b \le q$ by assumption, so we actually have $q = b$, and thus, putting $b = q$ in (7.39), we obtain

$$\big|q\xi - a\big| \le \big|q\xi - p\big|. \tag{7.40}$$

Now p, by definition, makes $|q\xi - x|$ the smallest over all x, so in particular, $|q\xi - p| \le |q\xi - a|$. Thus, by (7.40), we have

$$\big|q\xi - a\big| = \big|q\xi - p\big|.$$

Squaring both sides, we get

$$q^2\xi^2 - 2q\xi a + a^2 = q^2\xi^2 - 2q\xi p + p^2 \quad \Longrightarrow \quad 2q(p-a) = p^2 - a^2$$
$$\Longrightarrow \quad 2q\xi(p-a) = (p+a)(p-a) \quad \Longrightarrow \quad \xi = \frac{p+a}{2q}.$$

Here we can divide by $p - a$ because $a/b \neq p/q$ by assumption, and $b = q$, so $a \neq p$. In particular, $|a - p| \ge 1$ (since $a \neq p$ so $|a-p|$ is a positive integer), therefore

$$\big|q\xi - p\big| = \Big|\frac{p+a}{2q} - p\Big| = \Big|\frac{a-p}{2}\Big| \ge \frac{1}{2}. \tag{7.41}$$

We claim that $\xi = (p+a)/2q$ is reduced. Indeed, if the numerator and denominator had a common factor $m \ge 2$, then we could write $\xi = k/\ell$ where $p + a = mk$ and $2q = m\ell$. In particular, by (7.41), we would then have

$$\big|\ell\xi - k\big| = \Big|\ell\Big(\frac{k}{\ell}\Big) - k\Big| = 0 < \frac{1}{2} \quad \Longrightarrow \quad \big|\ell\xi - k\big| < \big|q\xi - p\big|. \tag{7.42}$$

We can have two choices $m = 2$ or $m > 2$. If $m = 2$, then $2q = 2\ell$, so $q = \ell$, and therefore substituting $\ell = q$ in (7.42), we have $|q\xi - k| < |q\xi - p|$. However, this contradicts the definition of p as the minimizer of $|q\xi - x|$ over all x. If $m > 2$, then $2\ell < m\ell = 2q$ implies that $\ell < q$. Now in view of the fact that $|\ell\xi - k| < |q\xi - p|$ in (7.42) and that $|q\xi - p| < |q_n\xi - p_n|$, we have $|\ell\xi - k| < |q_n\xi - p_n|$. However, the inequality $\ell < q$ contradicts the definition of q as the smallest positive denominator y in a rational x/y satisfying $|y\xi - x| < |q_n\xi - p_n|$. Thus, in summary, the assumption that p/q is not a best approximation leads us to the conclusion that $\xi = (p+a)/2q$ is reduced.

Now writing $\xi = (p+a)/2q$ into its canonical continued fraction expansion, if p_N/q_N denotes its last convergent, then we have $N > n+1$ and $\xi = \frac{p_N}{q_N}$, where

$p + a = p_N$ and $2q = q_N = a_N q_{N-1} + q_{N-2}$ with $a_N \geq 2$ because the expansion is canonical. Thus, by Theorem 7.18 and the inequality (7.41), we have

$$\left|q_{N-1}\xi - p_{N-1}\right| = q_{N-1}\left|\xi - \frac{p_{N-1}}{q_{N-1}}\right| < q_{N-1} \cdot \frac{1}{q_{N-1}q_N} = \frac{1}{q_N} = \frac{1}{2q} \leq \frac{1}{2} \leq \left|q\xi - p\right|.$$

In particular, since $|q\xi - p| < |q_n\xi - p_n|$ we have $|q_{N-1}\xi - p_{N-1}| < |q_n\xi - p_n|$. However, since $a_N \geq 2$ we have

$$2q_{N-1} \leq a_N q_{N-1} < a_N q_{N-1} + q_{N-2} = q_N = 2q \quad \Longrightarrow \quad q_{N-1} < q.$$

But this contradicts the definition of q as the smallest positive denominator y in a rational x/y satisfying $|y\xi - x| < |q_n\xi - p_n|$. This last contradiction shows that it is impossible that $\xi = (p + a)/2q$ is reduced, and hence, p/q must have been a best approximation to ξ; this finally completes our proof. □

Note that we left out c_0 in the statement of the theorem; this was intentional.

Example 7.24. Consider $\sqrt{3} = 1.73205080\ldots$. The best integer approximation to $\sqrt{3}$ is 2. In Subsection 7.4.3 we found that $\sqrt{3} = \langle 1; \overline{1, 2}\rangle$. Thus, $c_0 = 1$, which is not a best approximation. However, $c_1 = 1 + \frac{1}{1} = 2$ is a best approximation.

7.5.3. Dirichlet's approximation theorem. Using Theorem 7.20, we prove the following famous fact.

THEOREM 7.21 (**Dirichlet's approximation theorem**). *Amongst two consecutive convergents $p_n/q_n, p_{n+1}/q_{n+1}$ with $n \geq 0$ of the canonical continued fraction expansion to a real number (rational or irrational) ξ, one of them satisfies*

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{2q^2}. \tag{7.43}$$

Conversely, if a rational number p/q satisfies (7.43)*, then it is a convergent.*

PROOF. We begin by proving that a rational number satisfying (7.43) must be a convergent, then we show that convergents satisfy (7.43).

Step 1: Assume that p/q satisfies (7.43). To prove that it must be a convergent, we just need to show that it is a best approximation. To this end, assume that $a/b \neq p/q$ and that

$$\left|b\xi - a\right| \leq \left|q\xi - p\right|;$$

we must show that $q < b$. To prove this, we note that (7.43) implies that

$$\left|\xi - \frac{a}{b}\right| = \frac{1}{b}\left|b\xi - a\right| \leq \frac{1}{b}\left|q\xi - p\right| < \frac{1}{b} \cdot \frac{1}{2q} = \frac{1}{2bq}.$$

This inequality plus (7.43) give

$$\left|\frac{aq - bp}{bq}\right| = \left|\frac{a}{b} - \frac{p}{q}\right| = \left|\frac{a}{b} - \xi + \xi - \frac{p}{q}\right| \leq \left|\frac{a}{b} - \xi\right| + \left|\xi - \frac{p}{q}\right| < \frac{1}{2bq} + \frac{1}{2q^2}.$$

Since $a/b \neq p/q$, $|aq - bp|$ is a positive integer, that is, $1 \leq |aq - bp|$, therefore

$$\frac{1}{bq} < \frac{1}{2bq} + \frac{1}{2q^2} \quad \Longrightarrow \quad \frac{1}{2bq} < \frac{1}{2q^2} \quad \Longrightarrow \quad q < b.$$

We now show that one of two consecutive convergents satisfies (7.43). Let p_n/q_n and p_{n+1}/q_{n+1}, $n \geq 0$, be two consecutive convergents.

Step 2: Assume first that $q_n = q_{n+1}$. Since $q_{n+1} = a_{n+1}q_n + q_{n-1}$ we see that $q_n = q_{n+1}$ if and only if $n = 0$ (because $q_{n-1} = 0$ if and only if $n = 0$) and $a_1 = 1$, in which case, $q_1 = q_0 = 1$, $p_0 = a_0$, and $p_1 = a_0 a_1 + 1 = a_0 + 1$. Since $p_0/q_0 = a_0/1$ and $p_1/q_1 = (a_0 + 1)/1$, we have to show that

$$\big|\xi - a_0\big| < \frac{1}{2} \quad \text{or} \quad \big|\xi - (a_0 + 1)\big| < \frac{1}{2}.$$

But one of these must hold because $a_0 = \lfloor \xi \rfloor$, so

$$a_0 \le \xi < a_0 + 1.$$

Note that the special situation where ξ is exactly half-way between a_0 and $a_0 + 1$, that is, $\xi = a_0 + 1/2 = \langle a_0, 2 \rangle$, is not possible under our current assumptions because in this special situation, $q_1 = 2 \neq 1 = q_0$.

Step 3: Assume now that $q_n \neq q_{n+1}$. Consider two consecutive convergents c_n and c_{n+1}. We know that either

$$c_n < \xi < c_{n+1} \quad \text{or} \quad c_{n+1} < \xi < c_n,$$

depending on whether n is even or odd. For concreteness, assume that n is even; the odd case is entirely similar. Then from $c_n < \xi < c_{n+1}$ and the fundamental recurrence relation $c_{n+1} - c_n = 1/q_n q_{n+1}$, we see that

$$\big|\xi - c_n\big| + \big|c_{n+1} - \xi\big| = (\xi - c_n) + (c_{n+1} - \xi) = c_{n+1} - c_n = \frac{1}{q_n q_{n+1}}.$$

Now observe that since $q_n \neq q_{n+1}$, we have

$$0 < \frac{1}{2}\Big(\frac{1}{q_n} - \frac{1}{q_{n+1}}\Big)^2 = \frac{1}{2q_n^2} + \frac{1}{2q_{n+1}^2} - \frac{1}{q_n\, q_{n+1}} \quad \Longrightarrow \quad \frac{1}{q_n q_{n+1}} < \frac{1}{2q_n^2} + \frac{1}{2q_{n+1}^2},$$

so

$$\big|\xi - c_n\big| + \big|\xi - c_{n+1}\big| < \frac{1}{2q_n^2} + \frac{1}{2q_{n+1}^2}. \tag{7.44}$$

It follows that $|\xi - c_n| < 1/2q_n^2$ or $\big|\xi - c_{n+1}\big| < 1/2q_{n+1}$, otherwise (7.44) would fail to hold. This completes our proof. □

EXERCISES 7.5.

1. In this problem we find all the good approximations to $2/7$. First, to see things better, let's write down the some fractions with denominators less than 7:
$$\frac{0}{1} < \frac{1}{6} < \frac{1}{5} < \frac{1}{4} < \frac{2}{7} < \frac{1}{3} < \frac{2}{5} < \frac{1}{2}.$$
By examining the absolute values $\big|\xi - \frac{a}{b}\big|$ for the fractions listed, show that the good approximations to $2/7$ are $0/1, 1/2, 1/3, 1/4$, and of course, $2/7$. Now let's find which of the good approximations are best *without* using the best approximation theorem. To do so, compute the absolute values
$$\Big|1 \cdot \frac{2}{7} - 0\Big| \;,\; \Big|2 \cdot \frac{2}{7} - 1\Big| \;,\; \Big|3 \cdot \frac{2}{7} - 1\Big| \;,\; \Big|4 \cdot \frac{2}{7} - 1\Big|$$
and from these numbers, determine which of the good approximations are best. Using a similar method, find the good and best approximations to $3/7$, $3/5$, $8/5$, and $2/9$.
2. Prove that a real number ξ is rational if and only if there are infinitely many rational numbers p/q satisfying
$$\Big|\xi - \frac{p}{q}\Big| < \frac{1}{q^2}.$$
3. In this problem we find very beautiful approximations to π.

(a) Using the canonical continued fraction algorithm, prove that

$$\pi^4 = 97.40909103400242\ldots = \langle 97, 2, 2, 3, 1, 16539, 1, \ldots\rangle.$$

(Warning: If your calculator doesn't have enough decimal places of accuracy, you'll probably get a different value for 16539.)

(b) Compute $c_4 = \frac{2143}{22}$ and therefore, $\pi \approx \left(\frac{2143}{22}\right)^{1/4}$. Note that $\pi = 3.141592653\ldots$ while $(2143/22)^{1/4} = 3.141592652$, quite accurate! This approximation is due to Srinivasa Ramanujan (1887–1920) [**21**, p. 160].[4] As explained on Weinstein's website [**183**], we can write this approximation in **pandigital** form, that is, using all digits $0, 1, \ldots, 9$ exactly once :

$$\boxed{\pi \approx \left(\frac{2143}{22}\right)^{1/4} = \sqrt{\sqrt{0 + 3^4 + \frac{19^2}{78 - 56}}}.}$$

(c) By determining certain convergents of the continued fraction expansions of π^2, π^3, and π^5, derive the equally fascinating results:

$$\boxed{\pi \approx \sqrt{10}\ , \quad \left(\frac{227}{23}\right)^{1/2}\ , \quad 31^{1/3}\ , \quad \left(\frac{4930}{159}\right)^{1/3}\ , \quad 306^{1/5}\ , \quad \left(\frac{77729}{254}\right)^{1/5}\ .}$$

The approximation $\pi \approx \sqrt{10} = 3.162\ldots$ was known in Mesopotamia thousands of years before Christ [**130**]!

4. If $c_n = a_0 + \frac{b_1}{a_1} + \cdots + \frac{b_n}{a_n}$ and $\xi = a_0 + \frac{b_1}{a_1} + \frac{b_2}{a_2} + \cdots$, where $a_n \geq 1$ for $n \geq 1$, $b_n > 0$, and $\sum_{n=1}^{\infty} \frac{a_n a_{n+1}}{b_{n+1}} = \infty$, prove that for any $n = 0, 1, 2, \ldots$, we have $|\xi - c_{n+1}| < |\xi - c_n|$ and $|q_{n+1}\xi - p_{n+1}| < |q_n \xi - p_n|$ (cf. Theorem 7.18).

7.6. ★ Continued fractions and calendars, and math and music

We now do some fun stuff with continued fractions and their applications to calendars and pianos! In the exercises, you'll see how Christian Huygens (1629–1695), a Dutch physicist, made his model of the solar system (cf. [**115**]).

7.6.1. Calendars. Calendar making is an amazing subject; see Tøndering's (free!) book [**171**] for a fascinating look at calendars. A year, technically a **tropical year**, is the time it takes from one vernal equinox to the next. Recall that there are two equinoxes, which is basically (there is a more technical definition) the time when night and day have the same length. The vernal equinox occurs around March 21, the first day of spring, and the autumnal equinox occurs around September 23, the first day of fall. A year is approximately 365.24219 days. As you might guess, not being a whole number of days makes it quite difficult to make accurate calenders, and for this reason, the art of calendar making has been around since the beginning. Here are some approximations to a year that you might know about:

(1) 365 days, the ancient Egyptians and others.
(2) $365\frac{1}{4}$ days, Julius Caesar (100 B.C.–44 B.C.), 46 B.C., giving rise to the **Julian calendar**.
(3) $365\frac{97}{400}$ days, Pope Gregory XIII (1502–1585), 1585, giving rise to the **Gregorian calendar**, the calendar that is now the most widely-used calendar.

[4] *An equation means nothing to me unless it expresses a thought of God. Srinivasa Ramanujan (*1887–1920*).*

See Problem 1 for Persian calenders and their link to continued fractions. Let us analyze these more thoroughly. First, the ancient calendar consisting of 365 days is the basic calendar. Since a true year is 365.24219 days, an ancient year has

$$0.24219 \textit{ less} \text{ days than a true year.}$$

Thus, after 4 years, with an ancient calendar you'll lose

$$4 \times .24219 = 0.9687 \text{ days } \approx 1 \text{ day.}$$

After 125 years, with an ancient calendar you'll lose

$$125 \times .24219 = 30.27375 \text{ days } \approx 1 \text{ month.}$$

So, instead of having spring around March 21, you'll have it in February! After 500 years, with an ancient calendar you'll lose

$$500 \times .24219 = 121.095 \text{ days } \approx 4 \text{ months.}$$

So, instead of having spring around March 21, you'll have it in November! As you can see, this is getting quite ridiculous.

In the Julian calendar, there are an average of $365\frac{1}{4}$ days in a Julian year. The fraction $\frac{1}{4}$ is played out as we all know: We add *one* day to the ancient calendar every *four* years giving us a "leap year", that is, a year with 366 days. Thus, just as we said, a Julian calendar year gives the estimate

$$\frac{4 \times 365 + 1 \text{ days}}{4 \text{ years}} = 365\frac{1}{4}\ \frac{\text{days}}{\text{year}}.$$

The Julian year has

$$365.25 - 365.24219 = 0.00781 \textit{ more} \text{ days than a true year.}$$

So, for instance, after 125 years, with a Julian calendar you'll gain

$$125 \times .00781 = 0.97625 \text{ days } \approx 1 \text{ day.}$$

Not bad. After 500 years, with a Julian calendar you'll gain

$$500 \times .00781 = 3.905 \text{ days } \approx 4 \text{ days.}$$

Again, not bad! But, still, four days gained is still four days gained.

In the Gregorian calendar, there are an average of $365\frac{97}{400}$ days, that is, we add *ninety seven* days to the ancient calendar every *four hundred* years. These extra days are added as follows: Every four years we add one extra day, a "leap year" just like in the Julian calendar — however, this gives us 100 extra days in 400 years; so to offset this, we do not have a leap year for the century marks except 400, 800, 1200, 1600, 2000, 2400, ... multiples of 400. For example, consider the years

$$1604, 1608, \ldots, 1696, 1700, 1704, \ldots, 1796, 1800, 1804, \ldots, 1896, \\ 1900, 1904, \ldots, 1996, 2000.$$

Each of these years is a leap year except the three years 1700, 1800, and 1900 (but note that the year 2000 was a leap year since it is a multiple of 400, as you can verify on your old calendar). Hence, in the four hundred years from the end of 1600 to the end of 2000, we added only 97 total days since we didn't add extra days in 1700, 1800, and 1900. So, just as we said, a Gregorian calendar gives the estimate

$$\frac{400 \times 365 + 97}{400} = 365\frac{97}{400}\ \frac{\text{days}}{\text{year}}.$$

Since $365\frac{97}{400} = 365.2425$, the Gregorian year has

$$365.2425 - 365.24219 = 0.00031 \textit{ more} \text{ days than a true year.}$$

For instance, after 500 years, with a Gregorian calendar you'll gain

$$500 \times 0.00031 = 0.155 \text{ days } \approx 0 \text{ days!}$$

Now let's link calendars with continued fractions. Here is the continued fraction expansion of the tropical year:

$$365.24219 = \langle 365; 4, 7, 1, 3, 24, 6, 2, 2 \rangle.$$

This has convergents:

$$c_0 = 365\ ,\ c_1 = 365\frac{1}{4}\ ,\ c_2 = 365\frac{7}{29}\ ,\ c_3 = 365\frac{8}{33}\ ,\ c_4 = 365\frac{31}{128}\ ,\ldots.$$

Here, we see that c_0 is the ancient calendar and c_1 is the Julian calendar, but where is the Gregorian calendar? It's not on this list, but it's almost c_3 since

$$\frac{8}{33} = \frac{8}{33} \cdot \frac{12}{12} = \frac{96}{396} \approx \frac{97}{400}.$$

However, it turns out that $c_3 = 365\frac{8}{33}$ is *exactly* the average number of days in the Persian calendar introduced by the mathematician, astronomer, and poet Omar Khayyam (1048 –1131)! See Problem 1 for the modern Persian calendar!

7.6.2. Pianos. We now move from calendars to pianos. For more on the interaction between continued fractions and pianos, see [**49**], [**104**], [**12**], [**72**], [**7**], [**155**]. Let's start by giving a short lesson on music based on Euler's letter to a German princess [**30**] (see also [**82**]). When, say a piano wire or guitar string vibrates, it causes the air molecules around it to vibrate and these air molecules cause neighboring molecules to vibrate and finally, these molecules bounce against our ears, and we have the sensation of "sound". The rapidness of the vibrations, in number of vibrations per second, is called **frequency**. Let's say that we hear two notes with two different frequencies. In general, these frequencies mix together and don't produce a pleasing sound, but according to Euler, when the *ratio* of their frequencies happens to equal certain ratios of integers, then we hear a pleasant sound![5] Fascinating isn't it? We'll call the ratio of the frequencies an **interval** between the notes or the frequencies. For example, consider two notes, one with frequency f_1 and the other with frequency f_2 such that

$$\frac{f_2}{f_1} = \frac{2}{1} \quad \Longleftrightarrow \quad f_2 = 2f_1 \quad \text{(octave)};$$

in other words, the interval between the first and second note is 2, which is to say, f_2 is just twice f_1. This special interval is called an **octave**. It turns out that when two notes an octave apart are played at the same time, they sound beautiful together! Another interval that is corresponds to a beautiful sound is called the **fifth**, which is when the ratio is 3/2:

$$\frac{f_2}{f_1} = \frac{3}{2} \quad \Longleftrightarrow \quad f_2 = \frac{3}{2}f_1 \quad \text{(fifth)}.$$

[5] *Musica est exercitium arithmeticae occultum nescientis se numerare animi The pleasure we obtain from music comes from counting, but counting unconsciously. Music is nothing but unconscious arithmetic. From a letter to Goldbach, 27 April 1712, quoted in* [**153**].

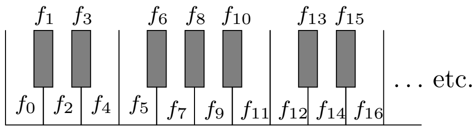

FIGURE 7.1. The k-th key, starting from $k = 0$, is labeled by its frequency f_k.

Other intervals (which remember just refer to ratios) that have names are

4/3 (fourth)	9/8 (major tone)	25/24 (chromatic semitone),
5/4 (major third)	10/9 (lesser tone)	81/80 (comma of Didymus),
6/5 (minor thirds)	16/15 (diatonic semitone).	

However, it is probably of universal agreement that the octave and the fifth make the prettiest sounds. Ratios such as 7/6, 8/7, 11/10, 12/11, ... don't seem to agree with our ears.

Now let's take a quick look at two facts concerning the piano. We all know what a piano keyboard looks like; see Figure 7.1. Let us label the (fundamental) frequencies of the piano keys, counting both white and black, by $f_0, f_1, f_2, f_3, \ldots$ starting from the far left key on the keyboard.[6] The first fact is that keys which are twelve keys apart are exactly an octave apart! For instance, f_0 and, jumping twelve keys to the right, f_{12} are an octave apart, f_7 and f_{19} are an octave apart, etc. For this reason, a piano scale really has just twelve basic frequencies, say $f_0, \ldots, f_{11}$, since by doubling these frequencies we get the twelve frequencies above, $f_{12}, \ldots, f_{23}$, and by doubling these we get $f_{24}, \ldots, f_{35}$, etc. The second fact is that a piano is **evenly tempered**, which means that the intervals between adjacent keys is constant. Let this constant be c. Then,

$$\frac{f_{n+1}}{f_n} = c \quad \Longrightarrow \quad f_{n+1} = cf_n$$

for all n. In particular,

$$f_{n+k} = cf_{n+k-1} = c(cf_{n+k-2}) = c^2 f_{n+k-2} = \cdots = c^k f_n. \tag{7.45}$$

Since $f_{n+12} = 2f_n$ (because f_n and f_{n+12} are an octave apart), it follows that with $k = 12$, we get

$$2f_n = c^{12} f_n \quad \Longrightarrow \quad 2 = c^{12} \quad \Longrightarrow \quad c = 2^{1/12}.$$

Thus, the interval between adjacent keys is $2^{1/12}$.

A question that might come to mind is: What is so special about the number twelve for a piano scale? Why not eleven or fifteen? Answer: It has to do with continued fractions! To see why, let us imagine that we have an evenly tempered piano with q basic frequencies, that is, keys that are q apart have frequencies differing by an octave. Question: Which q's make the best pianos? (Note: We better come up with $q = 12$ as one of the "best" ones!) By a very similar argument as we did above, we can see that the interval between adjacent keys is $2^{1/q}$. Now we have to ask: What makes a good piano? Well, our piano by design has octaves,

[6] A piano wire also gives off **overtones** but we focus here just on the fundamental frequency. Also, some of what we say here is not quite true for the keys near the ends of the keyboard because they don't vibrate well due of their stiffness leading to the phenomenon called **inharmonicity**.

but we would also like our piano to have fifths, the other beautiful interval. Let us label the keys of our piano as in Figure 7.1. Then we would like to have a p such that the interval between any frequency f_n and f_{n+p} is a fifth, that is,

$$\frac{f_{n+p}}{f_n} = \frac{3}{2}.$$

By the formula (7.45), which we can use in the present set-up as long as we put $c = 2^{1/q}$, we have $f_{n+p} = (2^{1/q})^p f_n = 2^{p/q} f_n$. Thus, we want

$$2^{p/q} = \frac{3}{2} \quad \Longrightarrow \quad \frac{p}{q} = \frac{\log(3/2)}{\log 2}.$$

This is, unfortunately, impossible because p/q is rational yet $\frac{\log(3/2)}{\log 2}$ is irrational (cf. Subsection 2.6.5)! Thus, it is impossible for our piano (even if $q = 12$ like our everyday piano) to have a fifth. However, hope is not lost because although our piano can never have a *perfect* fifth, it can certainly have an *approximate* fifth: We just need to find good rational approximations to the irrational number $\frac{\log(3/2)}{\log 2}$. This we know how to do using continued fractions. One can show that

$$\frac{\log(3/2)}{\log 2} = \langle 1, 1, 2, 2, 3, 1, \ldots \rangle,$$

which has convergents

$$0, \frac{1}{1}, \frac{1}{2}, \frac{3}{5}, \frac{7}{12}, \frac{24}{41}, \frac{31}{53}, \frac{179}{306}, \ldots.$$

Lo and behold, we see a twelve! In particular, by the best approximation theorem (Theorem 7.20), we know that 7/12 approximates $\frac{\log(3/2)}{\log 2}$ better than any rational number with a small denominator than twelve, which is to say, we cannot find a piano scale with fewer than twelve basic key that will give a better approximation to a fifth. This is why our everyday piano has twelve keys! In summary, $1, 2, 5, 12, 41, 53, 306, \ldots$ are the q's that make the "best" pianos. What about the other numbers in this list? Supposedly [**104**], in 40 B.C. King-Fang, a scholar of the Han dynasty, found the fraction 24/41, although to my knowledge, there has never been an instrument built with a scale of $q = 41$; however, King-Fang also found the fraction 31/53, and in this case, the $q = 53$ scale was advocated by Gerhardus Mercator (1512–1594) circa 1650 and was actually implemented by Robert Halford Macdowall Bosanquet (1841–1912) in his instrument *Enharmonic Harmonium* [**27**]!

We have focused on the interval of a fifth. What about other intervals? ... see Problem 2.

Exercises 7.6.

1. (**Persian calendar**) As of 2000, the modern calendar in Iran and Afghanistan has an average of $365\frac{683}{2820}$ days per year. The persian calendar introduced by Omar Khayyam (1048–1131) had an average of $365\frac{8}{33}$ days per year. Khayyam amazingly calculated the year to be 365.24219858156 days. Find the continued fraction expansion of 365.24219858156 and if $\{c_n\}$ are its convergents, show that c_0 is the ancient calendar, c_1 is the Julian calendar, c_3 is the calendar introduced by Khayyam, and c_7 is the modern Persian calendar!
2. Find the q's that will make a piano with the "best" approximations to a minor third. (Just as we found the q's that will make a piano with the "best" approximations to

fifth.) Do you see why many musicians, e.g. Aristoxenus, Kornerup, Ariel, Yasser, who enjoyed minor thirds, liked $q = 19$ musical scales? , , ,

3. (**A solar system model**) Christiaan Huygens (1629–1695) made a model scale of the solar system. In his day, it was thought that it took Saturn 29.43 years to make it once around the sun; that is,

$$\frac{\text{period of Saturn}}{\text{period of Earth}} = 29.43.$$

To make a realistic model of the solar system, Huygens needed to make gears for the model Saturn and the model Earth whose number of teeth had a ratio close to 29.43. Find the continued fraction expansion of 29.43 and see why Huygens chose the number of teeth to be 206 and 7, respectively. For more on the use of continued fractions to solve gear problems, see [**115**].

7.7. The elementary functions and the irrationality of $e^{p/q}$

In this section we derive some beautiful and classical continued fraction expansions for $\coth x$, $\tanh x$, and e^x. The book [**96**, Sec. 11.7] has a very nice presentation of this material.

7.7.1. The hypergeometric function. For complex $a \neq 0, -1, -2, \ldots$, the function

$$F(a, z) := 1 + \frac{1}{a} z + \frac{1}{a(a+1)} \frac{z^2}{2!} + \frac{1}{a(a+1)(a+2)} \frac{z^3}{3!} + \cdots, \quad z \in \mathbb{C},$$

is called a (simplified) **hypergeometric function** or more precisely, the **confluent hypergeometric limit function**. Using the ratio test, it is straightforward to check that $F(a, z)$ converges for all $z \in \mathbb{C}$. If for any $a \in C$, we define the **pochhammer symbol**, introduced by Leo August Pochhammer (1841–1920)

$$(a)_n := \begin{cases} 1 & n = 0 \\ a(a+1)(a+2)\cdots(a+n-1) & n = 1, 2, 3, \ldots, \end{cases}$$

then we can write the hypergeometric function in shorthand notation:

$$F(a, z) = \sum_{n=0}^{\infty} \frac{1}{(a)_n} \frac{z^n}{n!}.$$

Actually, the true hypergeometric function is defined by (cf. Subsection 5.3.4)

$$F(a, b, c, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n (c)_n} \frac{z^n}{n!},$$

but we won't need this function. Many familiar functions can be written in terms of these hypergeometric functions. For instance, consider

Proposition 7.22. *We have*

$$F\left(\frac{1}{2}, \frac{z^2}{4}\right) = \cosh z \quad , \quad z\, F\left(\frac{3}{2}, \frac{z^2}{4}\right) = \sinh z.$$

Proof. The proof of these identities are the same: We simply check that both sides have the same series expansions. For example, let us check the second identity; the identity for cosh is proved similarly. The function $z\, F\left(\frac{3}{2}, \frac{z^2}{4}\right)$ is just

$$z \cdot \sum_{n=0}^{\infty} \frac{1}{(3/2)_n} \frac{(z^2/2^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{(3/2)_n} \frac{z^{2n+1}}{2^{2n}\, n!},$$

and recall that

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}.$$

Thus, we just have to show that $(3/2)_n \, 2^{2n} \, n! = (2n+1)!$ for each n. Certainly this holds for $n = 0$. For $n \geq 1$, we have

$$\begin{aligned}(3/2)_n \, 2^{2n} \, n! &= \frac{3}{2}\left(\frac{3}{2}+1\right)\left(\frac{3}{2}+2\right)\cdots\left(\frac{3}{2}+n-1\right)\cdot 2^{2n} n! \\ &= \frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdots\frac{2n+1}{2}\cdot 2^{2n} n! \\ &= 3\cdot 5\cdot 7\cdots(2n+1)\cdot 2^n n!\end{aligned}$$

Since $2^n n! = 2^n \cdot 1 \cdot 2 \cdot 3 \cdots n = 2 \cdot 4 \cdot 6 \cdots 2n$, we have

$$\begin{aligned}3\cdot 5\cdot 7\cdots 2n+1\cdot 2^n n! &= 3\cdot 5\cdot 7\cdots(2n+1)\cdot 2\cdot 4\cdot 6\cdots 2n \\ &= 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdots 2n\cdot(2n+1) = (2n+1)!\end{aligned}$$

and our proof is complete. □

The hypergeometric function also satisfies an interesting, and useful as we'll see in a moment, recurrence relation.

PROPOSITION 7.23. *The hypergeometric function satisfies the following recurrence relation:*

$$F(a,z) = F(a+1,z) + \frac{z}{a(a+1)} F(a+2,z).$$

PROOF. The proof of this identity proceeds in the same way as in the previous proposition: We simply check that both sides have the same series expansions. We can write

$$F(a+1,z) + \frac{z}{a(a+1)} F(a+2,z) = \sum_{n=0}^{\infty} \frac{1}{(a+1)_n} \frac{z^n}{n!} + \sum_{n=0}^{\infty} \frac{1}{a(a+1)(a+2)_n} \frac{z^{n+1}}{n!}.$$

The constant term on the right is 1, which is the constant term on the left. For $n \geq 1$, coefficient of z^n on the right is

$$\begin{aligned}&\frac{1}{(a+1)_n \, n!} + \frac{1}{a(a+1)(a+2)_{n-1} \, (n-1)!} \\ &= \frac{1}{(a+1)\cdots(a+1+n-1)\, n!} + \frac{1}{a(a+1)\cdots(a+2+(n-1)-1)\,(n-1)!} \\ &= \frac{1}{(a+1)\cdots(a+n)\, n!} + \frac{1}{a(a+1)\cdots(a+n)\,(n-1)!} \\ &= \frac{1}{(a+1)\cdots(a+n)\,(n-1)!}\cdot\left(\frac{1}{n} + \frac{1}{a}\right) \\ &= \frac{1}{(a+1)\cdots(a+n)\,(n-1)!}\left(\frac{a+n}{a\cdot n}\right) \\ &= \frac{1}{a(a+1)\cdots(a+n-1)\, n(n-1)!} = \frac{1}{(a)_n \, n!},\end{aligned}$$

which is exactly the coefficient of z^n for $F(a,z)$. □

7.7.2. Continued fraction expansion of the hyperbolic cotangent. It turns out that Propositions 7.22 and 7.23 can be combined to give a fairly simple proof of the continued fraction expansion of the hyperbolic cotangent.

THEOREM 7.24. *For any real x, we have*

$$\boxed{\coth x = \cfrac{1}{x} + \cfrac{x}{3 + \cfrac{x^2}{5 + \cfrac{x^2}{7 + \cfrac{x^2}{9 + \ddots}}}}.}$$

PROOF. With $z = x > 0$, we have $F(a, x) > 0$ for any $a > 0$ by definition of the hypergeometric function. In particular, for $a > 0$, $F(a + 1, x) > 0$, so we can divide by this in Proposition 7.23, obtaining the recurrence relation

$$\frac{F(a,x)}{F(a+1,x)} = 1 + \frac{x}{a(a+1)} \frac{F(a+2,x)}{F(a+1,x)},$$

which we can write as

$$\frac{aF(a,x)}{F(a+1,x)} = a + \cfrac{x}{\cfrac{(a+1)F(a+1,x)}{F(a+2,x)}}.$$

Replacing a with $a + n$ with $n = 0, 1, 2, 3, \ldots$, we get

$$\frac{(a+n)F(a+n,x)}{F(a+n+1,x)} = a + n + \cfrac{x}{\cfrac{(a+n+1)F(a+n+1,x)}{F(a+n+2,x)}};$$

that is, if we define

$$\xi_n(a,x) := \frac{(a+n)F(a+n,x)}{F(a+n+1,x)} \quad , \quad a_n := a + n \ , \ b_n := x,$$

then

$$\xi_n(a,x) = a_n + \frac{b_{n+1}}{\xi_{n+1}(a,x)}, \quad n = 0, 1, 2, 3, \ldots. \tag{7.46}$$

Since

$$\sum_{n=1}^{\infty} \frac{a_n a_{n+1}}{b_n} = \sum_{n=1}^{\infty} \frac{(a+n)(a+n+1)}{x} = \infty,$$

by the continued fraction convergence theorem (Theorem 7.14), we know that

$$\frac{aF(a,x)}{F(a+1,x)} = \xi_0(a,x) = a + \frac{x}{a+1+} \frac{x}{a+2+} \frac{x}{a+3+} \frac{x}{a+4+} \frac{x}{a+5+} \cdots.$$

Since $F\left(1/2, x^2/4\right) = \cosh x$ and $x\,F\left(3/2, x^2/4\right) = \sinh x$ by Proposition 7.22, when we set $a = 1/2$ and replace x with $x^2/4$ into the previous continued fraction, we find

$$\frac{x \cosh x}{2 \sinh x} = \frac{x}{2} \coth x = \frac{1}{2} + \frac{x^2/4}{3/2\ +} \frac{x^2/4}{5/2\ +} \frac{x^2/4}{7/2\ +} \frac{x^2/4}{9/2\ +} \cdots,$$

or after multiplication by 2 and dividing by x, we get

$$\coth x = \frac{1}{x} + \frac{x/2}{3/2+} \frac{x^2/4}{5/2\ +} \frac{x^2/4}{7/2\ +} \frac{x^2/4}{9/2\ +} \cdots,$$

Finally, using the transformation rule (Theorem 7.1)

$$a_0 + \frac{b_1}{a_1+} \frac{b_2}{a_2+} \cdots \frac{b_n}{+\ a_n+} \cdots = a_0 + \frac{\rho_1 b_1}{\rho_1 a_1+} \frac{\rho_1\rho_2 b_2}{\rho_2 a_2\ +} \cdots + \frac{\rho_{n-1}\rho_n b_n}{\rho_n a_n\ \ +} \cdots$$

with $\rho_n = 2$ for all n, we get

$$\coth x = \frac{1}{x} + \frac{x}{3+} \frac{x^2}{5\ +} \frac{x^2}{7\ +} \frac{x^2}{9\ +} \cdots,$$

exactly what we set out to prove. □

Given any x, we certainly have $0 < b_n = x^2 < 2n+1 = a_n$ for all n sufficiently large, so by Theorem 7.15, it follows that when x is rational, $\coth x$ is irrational, or writing it out, for x rational,

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x}+1}{e^{2x}-1}$$

is irrational. It follows that for x rational, e^{2x} must be irrational too, for otherwise $\coth x$ would be rational contrary to assumption. Replacing x with $x/2$ and calling this r, we get the following neat corollary.

THEOREM 7.25. *e^r is irrational for any rational r.*

By the way, as did Johann Heinrich Lambert (1728–1777) originally did back in 1761 [**29**, p. 463], you can also use continued fractions to prove that π is irrational, see [**97**], [**120**]. As another easy corollary, we can get the continued fraction expansion for $\tanh x$. To do so, multiply the continued fraction for $\coth x$ by x:

$$x \coth x = b \quad , \quad \text{where } b = 1 + \frac{x^2}{3\ +} \frac{x^2}{5\ +} \frac{x^2}{7\ +} \frac{x^2}{9\ +} \cdots.$$

Thus, $\tanh x = \frac{x}{b}$, or replacing b with its continued fraction, we get

$$\boxed{\tanh x = \cfrac{x}{1 + \cfrac{x^2}{3 + \cfrac{x^2}{5 + \cfrac{x^2}{7 + \ddots}}}}.}$$

We derive one more beautiful expression that we'll need later. As before, we have

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x}+1}{e^{2x}-1} = \frac{1}{x} + \frac{x}{3+} \frac{x^2}{5\ +} \frac{x^2}{7\ +} \frac{x^2}{9\ +} \cdots.$$

Replacing x with $1/x$, we obtain

$$\frac{e^{2/x}+1}{e^{2/x}-1} = x + \frac{1/x}{3\ \ +} \frac{1/x^2}{5\ \ +} \frac{1/x^2}{7\ \ +} \frac{1/x^2}{9\ \ +} \cdots.$$

Finally, using the now familiar transformation rule, after a little algebra we get

$$\boxed{\frac{e^{2/x}+1}{e^{2/x}-1} = x + \cfrac{1}{3x + \cfrac{1}{5x + \cfrac{1}{7x + \ddots}}}}\,. \tag{7.47}$$

7.7.3. Continued fraction expansion of the exponential. We can now get the famous continued fraction expansion for e^x, which was first discovered by (as you might have guessed) Euler. To start, we observe that

$$\coth(x/2) = \frac{e^{x/2}+e^{-x/2}}{e^{x/2}+e^{-x/2}} = \frac{1+e^{-x}}{1-e^{-x}} \quad\Longrightarrow\quad e^{-x} = \frac{\coth(x/2)-1}{1+\coth(x/2)},$$

where we solved the equation on the left for e^{-x}. Thus,

$$e^{-x} = \frac{\coth(x/2)-1}{1+\coth(x/2)} = \frac{1+\coth(x/2)-2}{1+\coth(x/2)} = 1 - \frac{2}{1+\coth(x/2)},$$

so taking reciprocals, we get

$$e^x = \cfrac{1}{1 - \cfrac{2}{1+\coth(x/2)}},$$

By Theorem 7.24, we have

$$1+\coth(x/2) = 1 + \frac{2}{x} + \frac{x/2}{3\ +} \frac{x^2/4}{5\ +} \cdots = \frac{x+2}{x} + \frac{x/2}{3\ +} \frac{x^2/4}{5\ +} \frac{x^2/4}{7\ +} \cdots,$$

so

$$e^x = \frac{1}{1+} \frac{-2}{\frac{x+2}{x}+} \frac{x/2}{3\ +} \frac{x^2/4}{5\ +} \frac{x^2/4}{7\ +} \cdots$$

or using the transformation rule (Theorem 7.1)

$$\frac{b_1}{a_1+} \frac{b_2}{a_2+} \cdots \frac{b_n}{+\,a_n+} \cdots = \frac{\rho_1 b_1}{\rho_1 a_1+} \frac{\rho_1\rho_2 b_2}{\rho_2 a_2\ +} \cdots + \frac{\rho_{n-1}\rho_n b_n}{\rho_n a_n\ +} \cdots$$

with $\rho_1 = 1$, $\rho_2 = x$, and $\rho_n = 2$ for all $n \geq 3$, we get

$$e^x = \frac{1}{1+} \frac{-2x}{x+2+} \frac{x^2}{6\ +} \frac{x^2}{10+} \frac{x^2}{14+} \cdots.$$

Thus, we have derived Euler's celebrated continued fraction expansion for e^x:

THEOREM 7.26. *For any real x, we have*

$$\boxed{e^x = \cfrac{1}{1 - \cfrac{2x}{x+2+\cfrac{x^2}{6+\cfrac{x^2}{10+\cfrac{x^2}{14+\ddots}}}}}}.$$

In particular, if we let $x = 1$, we obtain

$$e = \cfrac{1}{1 - \cfrac{2}{3 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \ddots}}}}}.$$

Although beautiful, we can get an even more beautiful continued fraction expansion for e, which is a *simple* continued fraction.

7.7.4. The simple continued fraction expansion of e. If we expand the decimal number 2.718281828 into a simple continued fraction, we get (see Problem 2 in Exercises 7.4)

$$2.718281828 = \langle 2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1 \rangle.$$

For this reason, we should be able to conjecture that e should be

$$\boxed{e = \langle 2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, \ldots \rangle.} \tag{7.48}$$

This is true, and it was first proved by (as you might have guessed) Euler. Here,

$$a_0 = 2\ ,\ a_1 = 1\ ,\ a_2 = 2\ ,\ a_3 = 1\ ,\ a_4 = 1\ ,\ a_5 = 4\ ,\ a_6 = 1\ ,\ a_7 = 1,$$

and in general, for all $n \in \mathbb{N}$, $a_{3n-1} = 2n$ and $a_{3n} = a_{3n+1} = 1$. Since

$$2 = 1 + \cfrac{1}{0 + \cfrac{1}{1}},$$

we can write (7.48) in a prettier way that shows the full pattern:

$$\boxed{e = \langle 1; 0, 1, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, \ldots \rangle,} \tag{7.49}$$

or the somewhat more shockingly pretty

$$\boxed{e = 1 + \cfrac{1}{0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \ddots}}}}}}}.} \tag{7.50}$$

To prove this incredible formula, denote the convergents of the right-hand continued fraction in (7.48) by r_k/s_k. Since we have such simple relations $a_{3n-1} = 2n$ and $a_{3n} = a_{3n+1} = 1$ for all $n \in \mathbb{N}$, one might think that it is quite easy to compute formulas for r_{3n+1} and s_{3n+1}, and this thought is indeed the case.

LEMMA 7.27. *For all $n \geq 2$, we have*

$$r_{3n+1} = 2(2n+1)r_{3(n-1)+1} + r_{3(n-2)+1}$$
$$s_{3n+1} = 2(2n+1)s_{3(n-1)+1} + s_{3(n-2)+1}$$

PROOF. Both formulas are proved in similar ways, so we shall focus on the formula for r_{3n+1}. First, we apply our Wallis-Euler recursive formulas:

$$r_{3n+1} = r_{3n} + r_{3n-1} = \big(r_{3n-1} + r_{3n-2}\big) + r_{3n-1} = 2r_{3n-1} + r_{3n-2}.$$

We again apply the Wallis-Euler recursive formula on r_{3n-1}:

$$\begin{aligned} r_{3n+1} &= 2\Big(2nr_{3n-2} + r_{3n-3}\Big) + r_{3n-2} \\ &= \Big(2(2n)+1\Big)r_{3n-2} + 2r_{3n-3} \\ &= \Big(2(2n)+1\Big)r_{3n-2} + r_{3n-3} + r_{3n-3}. \end{aligned} \tag{7.51}$$

Again applying the Wallis-Euler recursive formula on the last term, we get

$$\begin{aligned} r_{3n+1} &= \Big(2(2n)+1\Big)r_{3n-2} + r_{3n-3} + \Big(r_{3n-4} + r_{3n-5}\Big) \\ &= \Big(2(2n)+1\Big)r_{3n-2} + \Big(r_{3n-3} + r_{3n-4}\Big) + r_{3n-5}. \end{aligned}$$

Since $r_{3n-2} = r_{3n-3} + r_{3n-4}$ by our Wallis-Euler recursive formulas, we finally get

$$\begin{aligned} r_{3n+1} &= \Big(2(2n)+1\Big)r_{3n-2} + r_{3n-2} + r_{3n-5} \\ &= \Big(2(2n)+2\Big)r_{3n-2} + r_{3n-5} \\ &= 2\Big((2n)+1\Big)r_{3(n-1)+1} + r_{3(n-2)+1}. \end{aligned}$$

□

Now putting $x = 1$ in (7.47), let us look at

$$\frac{e+1}{e-1} = \langle 2; 6, 10, 14, 18, \ldots \rangle.$$

that is, if the right-hand side is $\langle \alpha_0; \alpha_1, \ldots \rangle$, then $\alpha_n = 2(2n+1)$ for all $n = 0, 1, 2, \ldots$. If p_n/q_n are the convergents of this continued fraction, then we see that

$$p_n = 2(2n+1)p_{n-1} + p_{n-2} \quad \text{and} \quad q_n = 2(2n+1)q_{n-1} + q_{n-2},$$

which are similar to the relations in our lemma! Thus, it is not surprising in one bit that the r_{3n+1}'s and s_{3n+1}'s are related to the p_n's and q_n's. The exact relation is given in the following lemma.

LEMMA 7.28. *For all $n = 0, 1, 2, \ldots$, we have*

$$r_{3n+1} = p_n + q_n \quad \text{and} \quad s_{3n+1} = p_n - q_n.$$

PROOF. As with the previous lemma, we shall only prove the formula for r_{3n+1}. We proceed by induction: First, for $n = 0$, we have

$$r_1 := a_0 a_1 + 1 = 2 \cdot 1 + 1 = 3,$$

while $p_0 := 2$ and $q_0 := 1$, so $r_1 = p_0 + q_0$. If $n = 1$, then by the formula (7.51), we see that

$$r_{3\cdot 1+1} = (2(2)+1)r_1 + 2r_0 = 5 \cdot 3 + 2 \cdot 2 = 19.$$

On the other hand,

$$p_1 := \alpha_0\alpha_1 + 1 = 2\cdot 6 + 1 = 13 \quad , \quad q_1 := \alpha_1 = 6,$$

so $r_{3\cdot 1+1} = p_1 + q_1$.

Assume now that $r_{3k+1} = p_k + q_k$ for all $0 \le k \le n-1$ where $n \ge 2$; we shall prove that it holds for $k = n$ (this is an example of "strong induction"; see Section 2.2). But, by the Wallis-Euler recursive formulas, we have

$$\begin{aligned} r_{3n+1} &= 2(2n+1) r_{3(n-1)+1} + r_{3(n-2)+1} \\ &= 2(2n+1)(p_{n-1} + q_{n-2}) + (p_{n-2} + q_{n-2}) \\ &= 2(2n+1)p_{n-1} + p_{n-2} + 2(2n+1)q_{n-2} + q_{n-2} \\ &= p_n + q_n. \end{aligned}$$

□

Finally, we can now prove the continued fraction expansion for e:

$$\begin{aligned} \langle 2; 1, 1, 4, 1, 1, \ldots \rangle = \lim \frac{r_n}{s_n} = \lim \frac{r_{3n+1}}{s_{3n+1}} = \lim \frac{p_n + q_n}{p_n - q_n} \\ = \lim \frac{p_n/q_n + 1}{p_n/q_n - 1} = \frac{\frac{e+1}{e-1} + 1}{\frac{e+1}{e-1} - 1} = \frac{\frac{e}{e-1}}{\frac{1}{e-1}} = e. \end{aligned}$$

See Section 11.5 for Hermite's proof (cf. [**133**]). In the problems, we derive, along with other things, the following beautiful continued fraction for $\cot x$:

$$\text{(7.52)} \qquad \boxed{\cot x = \frac{1}{x} + \cfrac{x}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{7 - \cfrac{x^2}{9 - \ddots}}}}\;.}$$

From this continued fraction, we can derive the beautiful companion result for $\tan x$:

$$\boxed{\tan x = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{7 - \ddots}}}}\;.}$$

EXERCISES 7.7.

1. For all $n = 1, 2, \ldots$, let $a_n > 0$, $b_n \ge 0$, with $a_n \ge b_n + 1$. We shall prove that the following continued fraction converges:

$$\text{(7.53)} \qquad \frac{b_1}{a_1 +} \; \frac{-b_2}{a_2} \; + \; \frac{-b_3}{a_3} \; + \; \frac{-b_4}{a_4} \; + \; \cdots .$$

Note that for the continued fraction we are studying, $a_0 = 0$. Replacing b_n with $-b_n$ with $n \ge 2$ in the Wallis-Euler recurrence relations (7.16) and (7.17) we get

$$\begin{aligned} &p_n = a_n p_{n-1} - b_n p_{n-2} \quad , \quad q_n = a_n q_{n-1} - b_n q_{n-2}, \quad n = 2, 3, 4, \ldots \\ &p_0 = 0 \quad , \quad p_1 = b_1 \quad , \quad q_0 = 1 \quad , \quad q_1 = a_1. \end{aligned}$$

(i) Prove (via induction for instance) that $q_n \geq q_{n-1}$ for all $n = 1, 2, \ldots$. In particular, since $q_0 = 1$, we have $q_n \geq 1$ for all n, so the convergents $c_n = p_n/q_n$ of (7.53) are defined.

(ii) Verify that $q_1 - p_1 \geq 1 = q_0 - p_0$. Now prove by induction that $q_n - p_n \geq q_{n-1} - p_{n-1}$ for all $n = 1, 2, \ldots$. In particular, since $q_0 - p_0 = 1$, we have $q_n - p_n \geq 1$ for all n. Diving by q_n conclude that $0 \leq c_n \leq 1$ for all $n = 1, 2, \ldots$.

(iii) Using the fundamental recurrence relations for $c_n - c_{n-1}$, prove that $c_n - c_{n-1} \geq 0$ for all $n = 1, 2, \ldots$. Combining this with (1ii) shows that $0 \leq c_1 \leq c_2 \leq c_3 \leq \cdots \leq 1$; that is, $\{c_n\}$ is a bounded monotone sequence and hence converges. Thus, the continued fraction (7.53) converges.

2. For all $n = 1, 2, \ldots$, let $a_n > 0$, $b_n \geq 0$, with $a_n \geq b_n + 1$. From the previous problem, it follows that given any $a_0 \in \mathbb{R}$, the continued fraction $a_0 - \frac{b_1}{a_1 +} \frac{-b_2}{a_2 +} \frac{-b_3}{a_3 +} \frac{-b_4}{a_4 +} \cdots$ converges. We now prove a variant of the continued fraction convergence theorem (Theorem 7.14): Let $\xi_0, \xi_1, \xi_2, \ldots$ be any sequence of real numbers with $\xi_n > 0$ for $n \geq 1$ and suppose that these numbers are related by

$$\xi_n = a_n + \frac{-b_{n+1}}{\xi_{n+1}} \quad , \quad n = 0, 1, 2, \ldots.$$

Then ξ_0 is equal to the continued fraction

$$\xi_0 = a_0 - \frac{b_1}{a_1 +} \frac{-b_2}{a_2 +} \frac{-b_3}{a_3 +} \frac{-b_4}{a_4 +} \frac{-b_5}{a_5 +} \cdots.$$

Prove this statement following (almost verbatim!) the proof of Theorem 7.14.

3. We are now ready to derive the beautiful cotangent continued fraction (7.52).

(i) Let $a > 0$. Then as we derived the identity (7.46) found in Theorem 7.24, prove that if we define

$$\eta_n(a, x) := \frac{(a+n)F(a+n, -x)}{F(a+n+1, -x)} \quad , \quad a_n = a + n \ , \ b_n = x, \qquad n = 0, 1, 2, \ldots,$$

then

$$\eta_n(a, x) = a_n + \frac{-b_{n+1}}{\eta_{n+1}(a, x)}, \qquad n = 0, 1, 2, 3, \ldots.$$

(ii) Using Problem 2, prove that for $x \geq 0$ sufficiently small, we have

$$\frac{aF(a, -x)}{F(a+1, -x)} = \eta_0(a, x) = a - \frac{x}{a+1+} \frac{-x}{a+2+} \frac{-x}{a+3+} \frac{-x}{a+4+} \frac{-x}{a+5+} \cdots. \tag{7.54}$$

(iii) Prove that (cf. the proof of Proposition 7.22)

$$F\left(\frac{1}{2}, -\frac{x^2}{4}\right) = \cos x \quad , \quad z\, F\left(\frac{3}{2}, -\frac{x^2}{4}\right) = \sin x.$$

(iv) Now put $a = 1/2$ and replace x with $-x^2/4$ in (7.54) to derive the beautiful cotangent expansion (7.52). Finally, relax and contemplate this fine formula!

4. (**Irrationality of** $\log r$) Using Theorem 7.25, prove that if $r > 0$ is rational with $r \neq 1$, then $\log r$ is irrational. In particular, one of our favorite constants, $\log 2$, is irrational.

7.8. Quadratic irrationals and periodic continued fractions

We already know (Section 3.8) that a real number has a periodic decimal expansion if and only if the number is rational. One can ask the same thing about continued fractions: What types of real numbers have periodic continued fractions? The answer, as you will see in this section, are those real numbers called quadratic irrationals.

7.8.1. Periodic continued fractions. The object of this section is to characterize continued fractions that "repeat".

Example 7.25. We have already encountered the beautiful continued fraction

$$\frac{1+\sqrt{5}}{2} = \langle 1; 1, 1, 1, 1, 1, 1, 1, 1, \ldots \rangle.$$

We usually write the right-hand side as $\langle \overline{1} \rangle$ to emphasize that the 1 repeats.

Example 7.26. Another continued fraction that repeats is

$$\sqrt{8} = \langle 2; 1, 4, 1, 4, 1, 4, 1, 4, \ldots \rangle,$$

where we have an infinite repeating block of $1, 4$. We usually write the right-hand side as $\sqrt{8} = \langle 2; \overline{1, 4} \rangle$.

Example 7.27. Yet one more continued fraction that repeats is

$$\sqrt{19} = \langle 4; 2, 1, 3, 1, 2, 8, 2, 1, 3, 1, 2, 8, \ldots \rangle,$$

where we have an infinite repeating block of $2, 1, 3, 1, 2, 8$. We usually write the right-hand side as $\sqrt{19} = \langle 4; \overline{2, 1, 3, 1, 2, 8} \rangle$.

Notice that the above repeating continued fractions are continued fractions for expressions with square roots.

Example 7.28. Consider now the expression:

$$\xi = \langle 3; 2, 1, 2, 1, 2, 1, 2, 1, \ldots \rangle = \langle 3; \overline{2, 1} \rangle.$$

If $\eta = \langle 2; 1, 2, 1, 2, 1, 2, \ldots \rangle$, then $\xi = 3 + \frac{1}{\eta}$, and

$$\eta = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cdots}}} = 2 + \cfrac{1}{1 + \cfrac{1}{\eta}}.$$

Solving for η we find that $\eta = 1 + \sqrt{3}$. Hence,

$$\xi = 3 + \frac{1}{\eta} = 3 + \frac{1}{1+\sqrt{3}} = 3 + \frac{\sqrt{3}-1}{2} = \frac{5+\sqrt{3}}{6},$$

yet another square root expression.

Consider the infinite repeating continued fraction

$$\begin{aligned} (7.55) \quad \xi &= \langle a_0; a_1, \ldots, a_{\ell-1}, b_0, b_1, \ldots, b_{m-1}, b_0, b_1, \ldots, b_{m-1}, b_0, b_1, \ldots, b_{m-1}, \ldots \rangle \\ &= \langle a_0; a_1, \ldots, a_{\ell-1}, \overline{b_0, b_1, \ldots, b_{m-1}} \rangle, \end{aligned}$$

where the bar denotes that the block of numbers $b_0, b_1, \ldots, b_{m-1}$ repeats forever. Such a continued fraction is said to be **periodic**. When writing a continued fraction in this way we assume that there is no shorter repeating block and that the repeating block cannot start at an earlier position. For example, we would *never* write

$$\langle 2; 1, 2, 4, 3, 4, 3, 4, 3, 4, \ldots \rangle \quad \text{as} \quad \langle 2; 1, 2, 4, \overline{3, 4, 3, 4} \rangle;$$

we simply write it as $\langle 2; 1, 2, \overline{4, 3} \rangle$. The integer m is called the **period** of the continued fraction. An equivalent way to define a periodic continued fraction is as an infinite simple continued fraction $\xi = \langle a_0; a_1, a_2, \ldots\rangle$ such that for some m and ℓ, we have

$$a_n = a_{m+n} \quad \text{for all } n = \ell, \ell+1, \ell+2, \ldots. \tag{7.56}$$

The examples above suggest that infinite periodic simple continued fractions are intimately related to expressions with square roots; in fact, these expressions are called quadratic irrationals as we shall see in a moment.

7.8.2. Quadratic irrationals. A **quadratic irrational** is, exactly as its name suggests, an irrational real number that is a solution of a quadratic equation with integer coefficients. Using the quadratic equation, we leave you to show that a quadratic irrational ξ can be written in the form

$$\xi = r + s\sqrt{b} \tag{7.57}$$

where r, s are rational numbers and $b > 0$ is an integer that is not a perfect square (for if b were a perfect square, then $\sqrt{b}$ would be an integer so the right-hand side of ξ would be rational, contradicting that ξ is irrational). Conversely, given *any* real number of the form (7.57), one can check that ξ is a root of the equation

$$x^2 - 2r\,x + (r^2 - s^2 b) = 0.$$

Multiplying both sides of this equation by the common denominator of the rational numbers $2r$ and $r^2 - s^2b$, we can make the polynomial on the left have integer coefficients. Thus, a real number is a quadratic irrational if and only if it is of the form (7.57). As we shall see in Theorem 7.29 below, it would be helpful to write quadratic irrationals in a certain way. Let ξ take the form in (7.57) with $r = m/n$ and $s = p/q$ where we may assume that $n, q > 0$. Then with the help of some mathematical gymnastics, we see that

$$\xi = \frac{m}{n} + \frac{p\sqrt{b}}{q} = \frac{mq + np\sqrt{b}}{nq} = \frac{mq + \sqrt{bn^2p^2}}{nq} = \frac{mnq^2 + \sqrt{bn^4p^2q^2}}{n^2q^2}.$$

Notice that if we set $\alpha = mnq^2$, $\beta = n^2q^2$ and $d = bn^4p^2q^2$, then $d - \alpha^2 = bn^4p^2q^2 - m^2n^2q^4 = (bn^2p^2 - m^2q^2)(n^2q^2)$ is divisible by $\beta = n^2q^2$. Therefore, we can write any quadratic irrational in the form

$$\xi = \frac{\alpha + \sqrt{d}}{\beta}, \quad \alpha, \beta, d \in \mathbb{Z},\ d > 0 \text{ is not a perfect square, and } \beta \big| (d - \alpha^2).$$

Using this expression as the starting point, we prove the following nice theorem that gives formulas for the convergents of the continued fraction expansion of ξ.

Theorem 7.29. *The canonical simple continued fraction expansion of a quadratic irrational ξ has the complete quotients $\{\xi_n\}$ (with $\xi_0 = \xi$) and partial quotients $\{a_n\}$ determined by*

$$\xi_n = \frac{\alpha_n + \sqrt{d}}{\beta_n} \quad , \quad a_n = \lfloor \xi_n \rfloor,$$

where α_n and β_n are integers with $\beta_n > 0$ defined by the recursive sequences

$$\alpha_0 = \alpha\ ,\ \beta_0 = \beta\ ,\ \alpha_{n+1} = a_n\beta_n - \alpha_n\ ,\ \beta_{n+1} = \frac{d - \alpha_{n+1}^2}{\beta_n};$$

moreover, $\beta_n | (d - \alpha_n^2)$ for all n.

PROOF. We first show that all the α_n's and β_n's defined above are integers with β_n never zero and $\beta_n | (d - \alpha_n^2)$. This is automatic with $n = 0$. Assume this is true for n. Then $\alpha_{n+1} = a_n \beta_n - \alpha_n$ is an integer. To see that β_{n+1} is also an integer, observe that

$$\beta_{n+1} = \frac{d - \alpha_{n+1}^2}{\beta_n} = \frac{d - (a_n \beta_n - \alpha_n)^2}{\beta_n} = \frac{d - a_n^2 \beta_n^2 - 2a_n \beta_n \alpha_n - \alpha_n^2}{\beta_n}$$
$$= \frac{d - \alpha_n^2}{\beta_n} + 2a_n \alpha_n - a_n^2 \beta_n.$$

By induction hypothesis, $(d - \alpha_n^2)/\beta_n$ is an integer and so is $2a_n \alpha_n - a_n^2 \beta_n$. Thus, β_{n+1} is an integer too. Moreover, $\beta_{n+1} \neq 0$, because if $\beta_{n+1} = 0$, then we must have $d - \alpha_{n+1}^2 = 0$, which shows that d is a perfect square contrary to our condition on d. Finally, since β_n is an integer and

$$\beta_{n+1} = \frac{d - \alpha_{n+1}^2}{\beta_n} \quad \Longrightarrow \quad \beta_n = \frac{d - \alpha_{n+1}^2}{\beta_{n+1}},$$

β_{n+1} must divide $d - \alpha_{n+1}^2$.

Lastly, it remains to prove that the ξ_n's are the complete quotients of ξ (this automatically proves that the a_n's are the partial quotients because $a_n = \lfloor \xi_n \rfloor$ by definition). To do so, we simply use the formula for ξ_n:

$$\xi_n - a_n = \frac{\alpha_n + \sqrt{d}}{\beta_n} - \frac{\alpha_{n+1} + \alpha_n}{\beta_n} = \frac{\sqrt{d} - \alpha_{n+1}}{\beta_n}$$

where in the middle equality we solved $\alpha_{n+1} = a_n \beta_n - \alpha_n$ for a_n. Rationalizing and using the definition of β_{n+1} and ξ_{n+1}, we obtain

$$\xi_n - a_n = \frac{d - \alpha_{n+1}^2}{\beta_n(\sqrt{d} + \alpha_{n+1})} = \frac{\beta_{n+1}}{\sqrt{d} + \alpha_{n+1}} = \frac{1}{\xi_{n+1}} \quad \Longrightarrow \quad \xi_n = a_n + \frac{1}{\xi_{n+1}}.$$

This shows that the ξ_n's are the complete quotients of ξ (why?). □

7.8.3. Quadratic irrationals and periodic continued fractions. After one preliminary result, we shall prove that an infinite simple continued fraction is a quadratic irrational if and only if it is periodic. Define

$$\mathbb{Z}[\sqrt{d}] := \{a + b\sqrt{d}\,;\, a, b \in \mathbb{Z}\}$$

and

$$\mathbb{Q}[\sqrt{d}] := \{a + b\sqrt{d}\,;\, a, b \in \mathbb{Q}\}.$$

Given $\xi = a + b\sqrt{d}$ in either $\mathbb{Z}[\sqrt{d}]$ or $\mathbb{Q}[\sqrt{d}]$, we define its **conjugate** by

$$\overline{\xi} := a - b\sqrt{d}.$$

LEMMA 7.30. *$\mathbb{Z}[\sqrt{d}]$ is a commutative ring and $\mathbb{Q}[\sqrt{d}]$ is a field, and conjugation preserves the algebraic properties; for example, if $\alpha, \beta \in \mathbb{Q}[\sqrt{d}]$, then*

$$\overline{\alpha \pm \beta} = \overline{\alpha} \pm \overline{\beta}, \quad \overline{\alpha \cdot \beta} = \overline{\alpha} \cdot \overline{\beta}, \ \text{and} \ \ \overline{\alpha/\beta} = \overline{\alpha}/\overline{\beta}.$$

PROOF. To prove that $\mathbb{Z}[\sqrt{d}]$ is a commutative ring we just need to prove that it has the same algebraic properties as the integers in that $\mathbb{Z}[\sqrt{d}]$ is closed under addition, subtraction, and multiplication — for more on this definition see our discussion in Subsection 2.3.1. For example, to see that $\mathbb{Z}[\sqrt{d}]$ is closed under multiplication, let $\alpha = a + b\sqrt{d}$ and $\beta = a' + b'\sqrt{d}$ be elements of $\mathbb{Z}[\sqrt{d}]$; then,

$$\alpha\beta = (a + b\sqrt{d})(a' + b'\sqrt{d}) = aa' + bb'd + (ab' + a'b)\sqrt{d}, \tag{7.58}$$

which is also in $\mathbb{Z}[\sqrt{d}]$. Similarly, one can show that $\mathbb{Z}[\sqrt{d}]$ satisfies all the other properties of a commutative ring.

To prove that $\mathbb{Q}[\sqrt{d}]$ is a field we need to prove that it has the same algebraic properties as the rational numbers in that $\mathbb{Q}[\sqrt{d}]$ is closed under addition, multiplication, subtraction, and division (by nonzero elements) — for more on this definition see our discussion in Subsection 2.6.1. For example, to see that $\mathbb{Q}[\sqrt{d}]$ is closed under taking reciprocals, observe that if $\alpha = a + b\sqrt{d} \in \mathbb{Q}[\sqrt{d}]$ is not zero, then

$$\frac{1}{\alpha} = \frac{1}{a + b\sqrt{d}} \cdot \frac{a - b\sqrt{d}}{a - b\sqrt{d}} = \frac{a - b\sqrt{d}}{a^2 - b^2 d} = \frac{a}{a^2 - b^2 d} - \frac{b}{a^2 - b^2 d}\sqrt{d}$$

Note that $a^2 - b^2 d \neq 0$ since being zero would imply that $\sqrt{d} = a/b$ is rational, which by assumption we know is not. Similarly, one can show that $\mathbb{Q}[\sqrt{d}]$ satisfies all the other properties of a field.

Finally, we need to prove that conjugation preserves the algebraic properties. For example, let's prove the equality $\overline{\alpha \cdot \beta} = \overline{\alpha} \cdot \overline{\beta}$, leaving the other properties to you. If $\alpha = a + b\sqrt{d}$ and $\beta = a' + b'\sqrt{d}$, then according to (7.58), we have

$$\overline{\alpha\beta} = aa' + bb'd - (ab' + a'b)\sqrt{d}$$

But

$$\overline{\alpha}\overline{\beta} = (a - b\sqrt{d})(a' - b'\sqrt{d}) = aa' + bb'd - (ab' + a'b)\sqrt{d},$$

which equals $\overline{\alpha\beta}$. □

The following theorem was first proved by Joseph-Louis Lagrange (1736–1813).

THEOREM 7.31. *An infinite simple continued fraction is a quadratic irrational if and only if it is periodic.*

PROOF. We first prove the "if" part then the "only if" part.

Step 1: Let $\xi = \langle a_0; a_1, \ldots, a_{\ell-1}, \overline{b_0, \ldots, b_m}\rangle$ be periodic and define

$$\eta := \langle b_0; b_1, \ldots, b_m, b_0, b_1, \ldots, b_m, b_0, b_1, \ldots, b_m, \ldots\rangle = \langle b_0; b_1, \ldots, b_m, \eta\rangle,$$

so that $\xi = \langle a_0, a_1, \ldots, a_{\ell-1}, \eta\rangle$. Since $\eta = \langle b_0; b_1, \ldots, b_m, \eta\rangle$, by Theorem 7.4, we have

$$\eta = \frac{\eta s_{m-1} + s_{m-2}}{\eta t_{m-1} + t_{m-2}},$$

where s_n/t_n are the convergents for η. Multiplying both sides by $\eta t_{m-1} + t_{m-2}$, we see that

$$\eta^2 t_{m-1} + \eta t_{m-2} = \eta s_{m-1} + s_{m-2} \quad \Longrightarrow \quad a\,\eta^2 + b\,\eta + c = 0,$$

where $a = t_{m-1}$, $b = t_{m-2} - s_{m-1}$, and $c = -s_{m-2}$. Hence, η is a quadratic irrational. Now using that $\xi = \langle a_0, a_1, \ldots, a_{\ell-1}, \eta\rangle$ and Theorem 7.4, we obtain

$$\xi = \frac{\eta p_{m-1} + p_{m-2}}{\eta q_{m-1} + q_{m-2}},$$

where p_n/q_n are the convergents for ξ. Since η is a quadratic irrational, it follows that ξ is a quadratic irrational since $\mathbb{Q}[\sqrt{d}]$ is a field from Theorem 7.30. Thus, we have proved that periodic simple continued fractions are quadratic irrationals.

Step 2: Now let $\xi = \langle a_0; a_1, a_2, \ldots \rangle$ be a quadratic irrational; we shall prove that its continued fraction expansion is periodic. The trick to prove **Step 2** is to first show that the integers α_n and β_n of the complete quotients of ξ found in Theorem 7.29 are bounded. To implement this idea, let ξ_n be the n-th complete quotient of ξ. Then we can write $\xi = \langle a_0; a_1, a_2, \ldots, a_{n-1}, \xi_n \rangle$, so by Theorem 7.4 we have

$$\xi = \frac{\xi_n p_{n-1} + p_{n-2}}{\xi_n q_{n-1} + q_{n-2}}.$$

Solving for ξ_n, after a little algebra, we find that

$$-\xi_n = \frac{q_{n-2}}{q_{n-1}} \left(\frac{\xi - c_{n-2}}{\xi - c_{n-1}} \right).$$

Since conjugation preserves the algebraic operations by our lemma, we see that

$$-\overline{\xi}_n = \frac{q_{n-2}}{q_{n-1}} \left(\frac{\overline{\xi} - c_{n-2}}{\overline{\xi} - c_{n-1}} \right), \tag{7.59}$$

If $\xi = (\alpha + \sqrt{d})/\beta$, then $\overline{\xi} - \xi = 2\sqrt{d}/\beta \neq 0$. Therefore, since $c_k \to \xi$ as $k \to \infty$, it follows that as $n \to \infty$,

$$\left(\frac{\overline{\xi} - c_{n-2}}{\overline{\xi} - c_{n-1}} \right) \to \left(\frac{\overline{\xi} - \xi}{\overline{\xi} - \xi} \right) = 1.$$

In particular, there is a natural number N such that for $n > N$, $(\overline{\xi} - c_{n-2})/(\overline{\xi} - c_{n-1}) > 0$. Thus, as $q_k > 0$ for $k \geq 0$, according to (7.59), for $n > N$, we have $-\overline{\xi}_n > 0$. Hence, writing ξ_n, which is positive for $n \geq 1$, as $\xi_n = (\alpha_n + \sqrt{d})/\beta_n$ shown in Theorem 7.29, it follows that for $n > N$,

$$0 = 0 + 0 < \xi_n + (-\overline{\xi}_n) = 2\frac{\sqrt{d}}{\beta_n}.$$

So, for $n > N$, we have $\beta_n > 0$. Now solving the identity $\beta_{n+1} = \frac{d - \alpha_{n+1}^2}{\beta_n}$ in Theorem 7.29 for d we see that

$$\beta_n \beta_{n+1} + \alpha_{n+1}^2 = d.$$

For $n > N$, both β_n and β_{n+1} are positive, which implies that β_n and $|\alpha_n|$ cannot be too large; for instance, for $n > N$, we must have $0 < \beta_n \leq d$ and $0 \leq |\alpha_n| \leq d$. (For if either β_n or $|\alpha_n|$ were greater than d, then $\beta_n \beta_{n+1} + \alpha_{n+1}^2$ would be strictly larger than d, an impossibility since the sum is supposed to equal d.) In particular, if A is the finite set

$$A = \{(j, k) \in \mathbb{Z} \times \mathbb{Z}\,;\, -d \leq j \leq d\ ,\ 1 \leq k \leq d\},$$

then for the infinitely many $n > N$, the pair (α_n, β_n) is in the finite set A. By the pigeonhole principle, there must be distinct $i, j > N$ such that $(\alpha_j, \beta_j) = (\alpha_k, \beta_k)$. Assume that $j > k$ and let $m := j - k$. Then $j = m + k$, so

$$\alpha_k = \alpha_{m+k} \quad \text{and} \quad \beta_k = \beta_{k+m}.$$

Since $a_k = \lfloor \xi_k \rfloor$ and $a_{m+k} = \lfloor \xi_{m+k} \rfloor$, by Theorem 7.29 we have

$$\xi_k = \frac{\alpha_k + \sqrt{d}}{\beta_k} = \frac{\alpha_{m+k} + \sqrt{d}}{\beta_{m+k}} = \xi_{m+k} \quad \Longrightarrow \quad a_k = \lfloor \xi_k \rfloor = \lfloor \xi_{m+k} \rfloor = a_{m+k}.$$

Thus, using our formulas for α_{k+1} and β_{k+1} from Theorem 7.29, we see that

$$\alpha_{k+1} = a_k\beta_k - \alpha_k = a_{m+k}\beta_{m+k} - \alpha_{m+k} = \alpha_{m+k+1},$$

and

$$\beta_{k+1} = \frac{d - \alpha_{k+1}^2}{\beta_k} = \frac{d - \alpha_{m+k+1}^2}{\beta_{m+k}} = \beta_{m+k+1}.$$

Thus,

$$\xi_{k+1} = \frac{\alpha_{k+1} + \sqrt{d}}{\beta_{k+1}} = \frac{\alpha_{m+k+1} + \sqrt{d}}{\beta_{m+k+1}} = \xi_{m+k+1}$$
$$\Longrightarrow \quad a_{k+1} = \lfloor \xi_{k+1} \rfloor = \lfloor \xi_{m+k+1} \rfloor = a_{m+k+1}.$$

Continuing this process by induction shows that $a_n = a_{m+n}$ for all $n = k, k+1, k+2, k+3, \ldots$. Thus, by the definition of periodicity in (7.56), we see that ξ has a periodic simple continued fraction. □

A periodic continued fraction is called **purely periodic** if it is of the form $\xi = \langle \overline{a_0; a_1, \ldots, a_{m-1}} \rangle$.

Example 7.29. The simplest example of such a fraction is the golden ratio:

$$\Phi = \frac{1 + \sqrt{5}}{2} = \langle \overline{1} \rangle = \langle 1; 1, 1, 1, 1, 1, \ldots \rangle.$$

Observe that Φ has the following properties:

$$\Phi > 1 \quad \text{and} \quad \overline{\Phi} = \frac{1 - \sqrt{5}}{2} = -0.618\ldots \quad \Longrightarrow \quad \Phi > 1 \quad \text{and} \quad -1 < \overline{\Phi} < 0.$$

In the following theorem, Evariste Galois'[7] (1811–1832) first publication (at the age of 17), we characterize purely periodic expansions as those quadratic irrationals having these same properties. (Don't believe everything to read about the legendary Galois; see [**144**]. See [**167**] for an introduction to Galois' famous theory.)

Theorem 7.32. *A quadratic irrational ξ is purely periodic if and only if*

$$\xi > 1 \quad \textit{and} \quad -1 < \overline{\xi} < 0.$$

Proof. Assume that $\xi = \langle a_0; \ldots, a_{m-1}, a_0, a_1, \ldots, a_{m-1}, \ldots \rangle$ is purely periodic; we shall prove that $\xi > 1$ and $-1 < \overline{\xi} < 0$. Recall that in general, for any simple continued fraction, $\langle b_0; b_1, b_2, \ldots \rangle$ all the b_n's are positive after b_0. Thus, as a_0 appears again (and again, and again, ...) after the first a_0 in ξ, it follows that

[7] *[From the preface to his final manuscript (Evariste died from a pistol duel at the age of 20)] Since the beginning of the century, computational procedures have become so complicated that any progress by those means has become impossible, without the elegance which modern mathematicians have brought to bear on their research, and by means of which the spirit comprehends quickly and in one step a great many computations. It is clear that elegance, so vaunted and so aptly named, can have no other purpose. ... Go to the roots, of these calculations! Group the operations. Classify them according to their complexities rather than their appearances! This, I believe, is the mission of future mathematicians. This is the road on which I am embarking in this work. Evariste Galois (1811–1832).*

$a_0 \geq 1$. Hence, $\xi = a_0 + \frac{1}{\xi_1} > 1$. Now applying Theorem 7.4 to $\langle a_0; \ldots, a_{m-1}, \xi\rangle$, we get

$$\xi = \frac{\xi p_{m-1} + p_{m-2}}{\xi q_{m-1} + q_{m-2}},$$

where p_n/q_n are the convergents for ξ. Multiplying both sides by $\xi q_{m-1} + q_{m-2}$, we obtain

$$\xi^2 q_{m-1} + \xi q_{m-2} = \xi p_{m-1} + p_{m-2} \quad \Longrightarrow \quad f(\xi) = 0,$$

where $f(x) = q_{m-1}x^2 + (q_{m-2} - p_{m-1})x - p_{m-2}$ is a quadratic polynomial. In particular, ξ is a root of f. Taking conjugates, we see that

$$q_{m-1}\xi^2 + (q_{m-2} - p_{m-1})\xi - p_{m-2} = 0 \quad \Longrightarrow \quad q_{m-1}\overline{\xi}^2 + (q_{m-2} - p_{m-1})\overline{\xi} - p_{m-2} = 0,$$

therefore $\overline{\xi}$ is the other root of f. Now $\xi > 1$, thus $p_n > 0$, $p_n < p_{n+1}$, and $q_n < q_{n+1}$ for all n, so

$$f(-1) = (q_{m-1} - q_{m-2}) + (p_{m-1} - p_{m-2}) > 0 \quad \text{and} \quad f(0) = c = -p_{m-2} < 0.$$

By the intermediate value theorem $f(x) = 0$ for some $-1 < x < 0$. Since $\overline{\xi}$ is the other root of f we have $-1 < \overline{\xi} < 0$.

Assume now that ξ is a quadratic irrational with $\xi > 1$ and $-1 < \overline{\xi} < 0$; we shall prove that ξ is purely periodic. To do so, we first prove that if $\{\xi_n\}$ are the complete quotients of ξ, then $-1 < \overline{\xi}_n < 0$ for all n. Since $\xi_0 = \xi$, this is already true for $n = 0$ by assumption. Assume this holds for n; then,

$$\xi_n = a_n + \frac{1}{\xi_{n+1}} \quad \Longrightarrow \quad \frac{1}{\overline{\xi}_{n+1}} = \overline{\xi}_n - a_n < -a_n \leq -1 \quad \Longrightarrow \quad \frac{1}{\overline{\xi}_{n+1}} < -1.$$

The inequality $\frac{1}{\overline{\xi}_{n+1}} < -1$ shows that $-1 < \overline{\xi}_{n+1} < 0$ and completes the induction. Now we already know that ξ is periodic, so let us assume sake of contradiction that ξ is not purely periodic, that is, $\xi = \langle a_0; a_1, \ldots, a_{\ell-1}, \overline{a_\ell, \ldots, a_{\ell+m-1}}\rangle$ where $\ell \geq 1$. Then $a_{\ell-1} \neq a_{\ell+m-1}$ for otherwise we could start the repeating block at $a_{\ell-1}$, so

$$\xi_{\ell-1} = a_{\ell-1} + \langle\overline{a_\ell, \ldots, a_{\ell+m-1}}\rangle \neq a_{\ell+m-1} + \langle\overline{a_\ell, \ldots, a_{\ell+m-1}}\rangle = \xi_{\ell+m-1} \tag{7.60}$$

Observe that this expression shows that $\xi_{\ell-1} - \xi_{\ell+m-1} = a_{\ell-1} - a_{\ell+m-1}$ is an integer. In particular, taking conjugates, we see that

$$\overline{\xi}_{\ell-1} - \overline{\xi}_{\ell+m-1} = a_{\ell-1} - a_{\ell+m-1} = \xi_{\ell-1} - \xi_{\ell+m-1}.$$

Now we already proved that $-1 < \overline{\xi}_{\ell-1} < 0$, and $-1 < \overline{\xi}_{\ell+m-1} < 0$ which we write as $0 < -\overline{\xi}_{\ell+m-1} < 1$. Thus,

$$0 - 1 < \overline{\xi}_{\ell-1} + (-\overline{\xi}_{\ell+m-1}) < 0 + 1 \quad \Longrightarrow \quad -1 < \xi_{\ell-1} - \xi_{\ell+m-1} < 1,$$

since $\overline{\xi}_{\ell-1} - \overline{\xi}_{\ell+m-1} = \xi_{\ell-1} - \xi_{\ell+m-1}$. However, we noted that $\xi_{\ell-1} - \xi_{\ell+m-1}$ is an integer, and since the only integer strictly between -1 and 1 is 0, it must be that $\xi_{\ell-1} = \xi_{\ell+m-1}$. However, this contradicts (7.60), and our proof is complete. □

7.8.4. Square roots and periodic continued fractions. Recall that

$$\sqrt{19} = \langle 4; \overline{2, 1, 3, 1, 2, 8}\rangle;$$

if you didn't notice the beautiful symmetry before, observe that we can write this as $\sqrt{19} = \langle a_0; \overline{a_1, a_2, a_3, a_2, a_1, 2a_0}\rangle$ where the repeating block has a symmetric part and an ending part twice a_0. It turns that any square root has this nice symmetry property. To prove this fact, we first prove the following.

LEMMA 7.33. *If* $\xi = \langle \overline{a_0; a_1, \ldots, a_{m-1}} \rangle$ *is purely periodic, then* $-1/\overline{\xi}$ *is also purely periodic of the reversed form:* $-1/\overline{\xi} = \langle \overline{a_{m-1}; a_{m-2}, \ldots, a_0} \rangle$.

PROOF. Writing out the complete quotients $\xi, \xi_1, \xi_2, \ldots, \xi_{m-1}$ of

$$\xi = \langle \overline{a_0; a_1, \ldots, a_{m-1}} \rangle = \langle a_0; a_1, \ldots, a_{m-1}, \xi \rangle$$

we obtain

$$\xi = a_0 + \frac{1}{\xi_1}\ ,\ \xi_1 = a_1 + \frac{1}{\xi_2}\ , \ldots,\ \xi_{m-2} = a_{m-2} + \frac{1}{\xi_{m-1}}\ ,\ \xi_{m-1} = a_{m-1} + \frac{1}{\xi}.$$

Taking conjugates of all of these and listing them in reverse order, we find that

$$\frac{-1}{\overline{\xi}} = a_{m-1} - \overline{\xi}_{m-1}\ ,\ \frac{-1}{\overline{\xi}_{m-1}} = a_{m-2} - \overline{\xi}_{m-2}\ , \ldots,\ \frac{-1}{\overline{\xi}_2} = a_1 - \overline{\xi}_1\ ,\ \frac{-1}{\overline{\xi}_1} = a_0 - \overline{\xi}.$$

Let us define $\eta_0 := -1/\overline{\xi}$, $\eta_1 = -1/\overline{\xi}_{m-1}$, $\eta_2 = -1/\overline{\xi}_{m-2}, \ldots, \eta_{m-1} = -1/\overline{\xi}_1$. Then we can write the previous displayed equalities as

$$\eta_0 = a_{m-1} + \frac{1}{\eta_1}\ ,\ \eta_1 = a_{m-2} + \frac{1}{\eta_2}\ , \ldots\ , \eta_{m-2} = a_1 + \frac{1}{\eta_{m-1}}\ ,\ \eta_{m-1} = a_0 + \frac{1}{\eta_0};$$

in other words, η_0 is just the continued fraction:

$$\eta_0 = \langle a_{m-1}; a_{m-2}, \ldots, a_1, a_0, \eta_0 \rangle = \langle \overline{a_{m-1}; a_{m-2}, \ldots, a_1, a_0} \rangle.$$

Since $\eta_0 = -1/\overline{\xi}$, our proof is complete. □

Recall that the continued fraction expansion for $\sqrt{d}$ has the complete quotients ξ_n and partial quotients a_n determined by

$$\xi_n = \frac{\alpha_n + \sqrt{d}}{\beta_n}\quad,\quad a_n = \lfloor \xi_n \rfloor,$$

where the α_n, β_n's are *integers* given in Theorem 7.29. We are now ready to prove Adrien-Marie Legendre's (1752–1833) famous result.

THEOREM 7.34. *The simple continued fraction of* $\sqrt{d}$ *has the form*

$$\sqrt{d} = \langle a_0; \overline{a_1, a_2, a_3, \ldots, a_3, a_2, a_1, 2a_0} \rangle.$$

Moreover, $\beta_n \neq -1$ *for all* n*, and* $\beta_n = +1$ *if and only if* n *is a multiple of the period of* $\sqrt{d}$.

PROOF. Starting the continued fraction algorithm for $\sqrt{d}$, we obtain $\sqrt{d} = a_0 + \frac{1}{\xi_1}$, where $\xi_1 > 1$. Since $\frac{1}{\xi_1} = -a_0 + \sqrt{d}$, we have

$$-\frac{1}{\overline{\xi}_1} = -\big(-a_0 - \sqrt{d}\big) = a_0 + \sqrt{d} > 1, \tag{7.61}$$

so we must have $-1 < \overline{\xi}_1 < 0$. Since both $\xi_1 > 1$ and $-1 < \overline{\xi}_1 < 0$, by Galois' Theorem 7.32, we know that ξ_1 is purely periodic: $\xi_1 = \langle \overline{a_1; a_2, \ldots, a_m} \rangle$. Thus,

$$\sqrt{d} = a_0 + \frac{1}{\xi_1} = \langle a_0; \xi_1 \rangle = \langle a_0; \overline{a_1; a_2, \ldots, a_m} \rangle.$$

On the other hand, from (7.61) and from Lemma 7.33, we see that

$$\begin{aligned}\langle 2a_0; a_1, a_2, \ldots, a_m, a_1, a_2, \ldots, a_m, \ldots \rangle &= a_0 + \sqrt{d} = -\frac{1}{\overline{\xi}_1} = \langle \overline{a_m; \ldots, a_1} \rangle \\ &= \langle a_m, a_{m-1}, a_{m-2}, \ldots, a_1, a_m, a_{m-1}, a_{m-2}, \ldots, a_1, \ldots \rangle.\end{aligned}$$

Comparing the left and right-hand sides, we see that $a_m = 2a_0$, $a_{m-1} = a_1$, $a_{m-2} = a_2$, $a_{m-3} = a_3$, and so forth, therefore,

$$\sqrt{d} = \langle a_0; \overline{a_1, a_2, \ldots, a_m} \rangle = \langle a_0; \overline{a_1, a_2, a_3, \ldots, a_3, a_2, a_1, 2a_0} \rangle.$$

We now prove that β_n never equals -1, and $\beta_n = +1$ if and only if n is a multiple of the period m. By the form of the continued fraction expansion of $\sqrt{d}$ we just derived, observe that for any $n > 0$, the n-th complete quotient ξ_n for $\sqrt{d}$ is purely periodic. In particular, by Galois' Theorem 7.32 we know that

$$n > 1 \implies \xi_n > 1 \quad \text{and} \quad -1 < \overline{\xi}_n < 0. \tag{7.62}$$

Now for sake of contradiction, assume that $\beta_n = -1$. Since $\beta_0 = +1$ by definition (see Theorem 7.29), we must have $n > 0$. Then the formula $\xi_n = (\alpha_n + \sqrt{d})/\beta_n$ with $\beta_n = -1$ and (7.62) imply that

$$1 < \xi_n = -\alpha_n - \sqrt{d} \implies \alpha_n < -1 - \sqrt{d} \implies \alpha_n < 0.$$

On the other hand, (7.62) also implies that

$$-1 < \overline{\xi}_n = -\alpha_n + \sqrt{d} < 0 \implies \sqrt{d} < \alpha_n \implies 0 < \alpha_n.$$

Since $\alpha_n < 0$ and $\alpha_n > 0$ cannot possibly hold, it follows that $\beta_n = -1$ is impossible.

We now prove that $\beta_n = +1$ if and only if n is a multiple of the period m. Assume first that $\beta_n = 1$. Then $\xi_n = \alpha_n + \sqrt{d}$. By (7.62) we see that

$$-1 < \overline{\xi}_n = \alpha_n - \sqrt{d} < 0 \implies \sqrt{d} - 1 < \alpha_n < \sqrt{d}.$$

Since α_n is an integer, and the only integer strictly between $\sqrt{d} - 1$ and $\sqrt{d}$ is $a_0 = \lfloor \sqrt{d} \rfloor$, it follows that $\alpha_n = a_0$, so $\xi_n = a_0 + \sqrt{d}$. Now recalling the expansion $\sqrt{d} = \langle a_0; \overline{a_1, a_2, \ldots, a_m} \rangle$ and the fact that $2a_0 = a_m$, it follows that

$$\begin{aligned} a_0 + \sqrt{d} &= \langle 2a_0, a_1, a_2, \ldots, a_{m-1}, a_m, a_1, a_2, \ldots, a_{m-1}, a_m, \ldots \rangle \\ &= \langle \overline{a_m; a_1, a_2, \ldots, a_{m-1}} \rangle; \end{aligned} \tag{7.63}$$

thus $\xi_n = \langle \overline{a_m; a_1, a_2, \ldots, a_{m-1}} \rangle$. On the other hand, ξ_n is by definition the n-th convergent of

$$\sqrt{d} = \langle a_0; a_1, a_2, \ldots, a_m, a_1, a_2, \ldots, a_m, \ldots \rangle,$$

so writing $n = mj + \ell$ where $j = 0, 1, 2, \ldots$ and $1 \le \ell \le m$, going out n slots after a_0, we see that

$$\xi_n = \langle \overline{a_\ell; a_{\ell+1}, a_{\ell+2}, \ldots, a_m, a_1, \ldots, a_{\ell-1}} \rangle.$$

Comparing this with $\xi_n = \langle \overline{a_m; a_1, a_2, \ldots, a_{m-1}} \rangle$, we must have $\ell = m$, so $n = mj + m = m(j+1)$ is a multiple of m.

Assume now that n is a multiple of m; say $n = mk$. Then going out $n = mk$ slots to the right of a_0 in the continued fraction expansion of $\sqrt{d}$ we get $\xi_n = \langle \overline{a_m; a_1, a_2, \ldots, a_{m-1}} \rangle$. Thus, $\xi_n = a_0 + \sqrt{d}$ by (7.63). Since $\xi_n = (\alpha_n + \sqrt{d})/\beta_n$ also, it follows that $\beta_n = 1$ and our proof is complete. □

Exercises 7.8.

1. Find the canonical continued fraction expansions for

$$(a)\ \sqrt{29} \quad , \quad (b)\ \frac{1 + \sqrt{13}}{2} \quad , \quad (c)\ \frac{2 + \sqrt{5}}{3}.$$

2. Find the values of the following continued fractions:

$$(a)\ \langle 3; \overline{2, 6} \rangle \quad , \quad (b)\ \langle \overline{1; 2, 3} \rangle \quad , \quad (c)\ \langle 1; 2, \overline{3} \rangle \quad , \quad (d)\ \langle 2; 5, \overline{1, 3, 5} \rangle.$$

3. Let $m, n \in \mathbb{N}$. Find the quadratic irrational numbers represented by

$$(a)\ \langle \overline{n} \rangle = \langle n; n, n, n, \ldots \rangle \quad , \quad (b)\ \langle \overline{n; 1} \rangle \quad , \quad (c)\ \langle \overline{n; n+1} \rangle \quad , \quad (d)\ \langle m; \overline{n} \rangle.$$

7.9. Archimedes' crazy cattle conundrum and diophantine equations

Archimedes of Syracuse (287–212) was known to think in preposterous proportions. In *The Sand Reckoner* [**123**, p. 420], a fun story written by Archimedes, he concluded that if he could fill the universe with grains of sand, there would be (less than) 8×10^{63} grains! According to Pappus of Alexandria (290–350), at one time Archimedes said (see [**46**, p. 15]) "Give me a place to stand on, and I will move the earth!" In the following we shall look at a cattle problem proposed by Archimedes, whose solution involves approximately 8×10^{206544} cattle! If you feel mooooooved to read more on Achimedes' cattle, see [**121**], [**177**], [**17**], [**186**], and [**105**].

7.9.1. Archimedes' crazy cattle conundrum. Here is a poem written by Archimedes to students at Alexandria in a letter to Eratosthenes of Cyrene (276 BC–194 BC). (The following is adapted from [**76**], as written in [**17**].)

> Compute, O stranger! the number of cattle of Helios, which once grazed on the plains of Sicily, divided according to their color, to wit:
> (1) White bulls = $\frac{1}{2}$ black bulls + $\frac{1}{3}$ black bulls + yellow bulls
> (2) Black bulls = $\frac{1}{4}$ spotted bulls + $\frac{1}{5}$ spotted bulls + yellow bulls
> (3) spotted bulls = $\frac{1}{6}$ white bulls + $\frac{1}{7}$ white bulls + yellow bulls
> (4) White cows = $\frac{1}{3}$ black herd + $\frac{1}{4}$ black herd (here, "herd" = bulls + cows)
> (5) Black cows = $\frac{1}{4}$ spotted herd + $\frac{1}{5}$ spotted herd
> (6) Dappled cows = $\frac{1}{5}$ yellow herd + $\frac{1}{6}$ yellow herd
> (7) Yellow cows = $\frac{1}{6}$ white herd + $\frac{1}{7}$ white herd
>
> He who can answer the above is no novice in numbers. Nevertheless he is not yet skilled in wise calculations! But come consider also all the following numerical relations between the Oxen of the Sun:
> (8) If the white bulls were combined with the black bulls they would be in a figure equal in depth and breadth and the far stretching plains of Sicily would be covered by the square formed by them.
> (9) Should the yellow and spotted bulls were collected in one place, they would stand, if they ranged themselves one after another, completing the form of an equilateral triangle.
>
> If thou discover the solution of this at the same time; if thou grasp it with thy brain; and give correctly all the numbers; O Stranger! go and exult as conqueror; be assured that thou art by all means proved to have abundant of knowledge in this science.

To solve this puzzle, we need to turn it into mathematics! Let W, X, Y, Z denote the number of white, black, yellow, and spotted bulls, respectively, and w, x, y, z for the number of white, black, yellow, and spotted cows, respectively.

The the conditions (1) – (7) can be written as

$$(1)\ W = \Big(\frac{1}{2} + \frac{1}{3}\Big)X + Y \qquad\qquad (2)\ X = \Big(\frac{1}{4} + \frac{1}{5}\Big)Z + Y$$

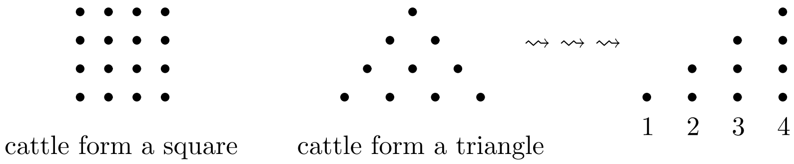

FIGURE 7.2. With the dots as bulls, on the left, the number of bulls is a square number (4^2 in this case) and the number of bulls on the right is a triangular number ($1 + 2 + 3 + 4$ in this case).

$$(3)\ Z = \left(\frac{1}{6} + \frac{1}{7}\right) W + Y \qquad (4)\ w = \left(\frac{1}{3} + \frac{1}{4}\right)(X + x)$$
$$(5)\ x = \left(\frac{1}{4} + \frac{1}{5}\right)(Z + z) \qquad (6)\ z = \left(\frac{1}{5} + \frac{1}{6}\right)(Y + y)$$
$$(7)\ y = \left(\frac{1}{6} + \frac{1}{7}\right)(W + w).$$

Now how do we interpret (8) and (9)? We will interpret (8) as meaning that the number of white and black bulls should be a square number (a perfect square); see the left picture in Figure 7.2. A **triangular number** is a number of the form

$$1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2},$$

for some n. Then we will interpret (9) as meaning that the number of yellow and spotted bulls should be a triangular number; see the right picture in Figure 7.2. Thus, (8) and (9) become

$$(8)\ W + X = \text{a square number} \quad , \qquad (9)\ Y + Z = \text{a triangular number}.$$

In summary: We want to find *integers* W, X, Y, Z, w, x, y, z (here we assume there are no such thing as "fractional cattle") solving equations (1)–(9). Now to the solution of Archimedes cattle problem. First of all, equations (1)–(7) are just linear equations so these equations can be solved using simple linear algebra. Instead of solving these equations by hand, which will probably take a few hours, it might be best to use the computer. Doing so you will find that in order for W, X, Y, Z, w, x, y, z to solve (1)–(7), they must be of the form

$$(7.64)\quad \begin{aligned} &W = 10366482\,k \ , \ X = 7460514\,k \ , \ Y = 4149387\,k \ , \ Z = 7358060\,k \\ &w = 7206360\,k \ , \ x = 4893246\,k \ , \ y = 5439213\,k \ , \ z = 3515820\,k, \end{aligned}$$

where k can equal $1, 2, 3, \ldots$. Thus, in order for us to fulfill conditions (1)–(7), we would have at the very least, setting $k = 1$,

$$\begin{aligned} &10366482 + 7460514 + 4149387 + 7358060 + 7206360 + 4893246 \\ &\qquad + 5439213 + 3515820 = 50389082 \approx 50 \text{ million cattle!} \end{aligned}$$

Now we are "no novice in numbers!" Nevertheless we are not yet skilled in wise calculations! To be skilled, we still have to satisfy conditions (8) and (9). For (8), this means

$$W + X = 10366482\,k + 7460514\,k = 17826996\,k = \text{a square number}.$$

Factoring $17826996 = 2^2 \cdot 3 \cdot 11 \cdot 29 \cdot 4657$ into its prime factors, we see that we must have

$$2^2 \cdot 3 \cdot 11 \cdot 29 \cdot 4657\,k = (\cdots)^2,$$

a square of an integer. Thus, we need $3 \cdot 11 \cdot 29 \cdot 4657\, k$ to be a square, which holds if and only if

$$k = 3 \cdot 11 \cdot 29 \cdot 4657\, m^2 = 4456749\, m^2$$

for some integer m. Plugging this value into (7.64), we get

$$\begin{aligned} W &= 46200808287018\, m^2 \quad , \quad X = 33249638308986\, m^2 \\ Y &= 18492776362863\, m^2 \quad , \quad Z = 32793026546940\, m^2 \\ w &= 32116937723640\, m^2 \quad , \quad x = 21807969217254\, m^2 \\ y &= 24241207098537\, m^2 \quad , \quad z = 15669127269180\, m^2, \end{aligned} \tag{7.65}$$

where m can equal $1, 2, 3, \ldots$. Thus, in order for us to fulfill conditions (1)–(8), we would have at the very least, setting $m = 1$,

$$\begin{aligned} &46200808287018 + 33249638308986 + 18492776362863 + 32793026546940 \\ &\quad + 32116937723640 + 21807969217254 + 24241207098537 \\ &\quad\quad + 15669127269180 = 2.2457\ldots \times 10^{14} \approx 2.2 \text{ trillion cattle!} \end{aligned}$$

It now remains to satisfy condition (9):

$$\begin{aligned} Y + Z &= 18492776362863\, m^2 + 32793026546940\, m^2 \\ &= 51285802909803\, m^2 = \frac{\ell(\ell+1)}{2}, \end{aligned}$$

for some integer ℓ. Multiplying both sides by 8 and adding 1, we obtain

$$8 \cdot 51285802909803\, m^2 + 1 = 4\ell^2 + 4\ell + 1 = (2\ell+1)^2 = n^2,$$

where $n = 2\ell + 1$. Since $8 \cdot 51285802909803 = 410286423278424$, we finally conclude that conditions (1)–(9) are all fulfilled if we can find *integers* m, n satisfying the equation

$$n^2 - 410286423278424\, m^2 = 1. \tag{7.66}$$

This is commonly called a **Pell equation** and is an example of a diophantine equation. As we'll see in the next subsection, we can solve this equation by simply (!) finding the simple continued fraction expansion of $\sqrt{410286423278424}$. The calculations involved are just sheer madness, but they can be done and have been done [**17**], [**186**]. In the end, we find that the smallest total number of cattle which satisfy (1)–(9) is a number with 206545 digits (!) and is equal to

$$7760271406\ldots(206525 \text{ other digits go here})\ldots 9455081800 \approx 8 \times 10^{206544}.$$

We are now skilled in wise calculations! A copy of this number is printed on 42 computer sheets and has been deposited in the Mathematical Tables of the journal *Mathematics of Computation* if you are interested.

7.9.2. Pell's equation. Generalizing the cattle equation (7.66), we call a diophantine equation of the form

$$x^2 - d\, y^2 = 1 \tag{7.67}$$

a **Pell equation**. Note that $(x, y) = (1, 0)$ solves this equation. This solution is called the **trivial solution**; the other solutions are not so easily attained. We remark that Pell's equation was named by Euler after John Pell (1611–1685), although Brahmagupta[8] (598–670) studied this equation a thousand years earlier [**29**, p. 221]. Any case, we shall see that the continued fraction expansion of $\sqrt{d}$ plays an important role in solving this equation. We note that if (x, y) solves (7.67), then trivially so do $(\pm x, \pm y)$ because of the squares in (7.67); thus, we usually restrict ourselves to the positive solutions.

Recall that the continued fraction expansion for $\sqrt{d}$ has the complete quotients ξ_n and partial quotients a_n determined by

$$\xi_n = \frac{\alpha_n + \sqrt{d}}{\beta_n} \quad , \quad a_n = \lfloor \xi_n \rfloor,$$

where α_n and β_n are integers defined in Theorem 7.29. The exact forms of these integers are not important; what is important is that β_n never equals -1 and $\beta_n = +1$ if and only if n is a multiple of the period of $\sqrt{d}$ as we saw in Theorem 7.34. The following lemma shows how the convergents of $\sqrt{d}$ enter Pell's equation.

LEMMA 7.35. *If p_n/q_n denotes the n-th convergent of $\sqrt{d}$, then for all $n = 0, 1, 2, \ldots$, we have*

$$p_n^2 - d\, q_n^2 = (-1)^{n+1} \beta_{n+1}.$$

PROOF. Since we can write $\sqrt{d} = \langle a_0; a_1, a_2, a_3, \ldots, a_n, \xi_{n+1} \rangle$ and $\xi_{n+1} = (\alpha_{n+1} + \sqrt{d})/\beta_{n+1}$, by (7.19) of Corollary 7.6, we have

$$\sqrt{d} = \frac{\xi_{n+1} p_n + p_{n-1}}{\xi_{n+1} q_n + q_{n-1}} = \frac{(\alpha_{n+1} + \sqrt{d})\, p_n + \beta_{n+1} p_{n-1}}{(\alpha_{n+1} + \sqrt{d})\, q_n + \beta_{n+1} q_{n-1}}.$$

Multiplying both sides by the denominator of the right-hand side, we get

$$\begin{aligned} &\sqrt{d}(\alpha_{n+1} + \sqrt{d})\, q_n + \sqrt{d} \beta_{n+1} q_{n-1} = (\alpha_{n+1} + \sqrt{d})\, p_n + \beta_{n+1} p_{n-1} \\ &\qquad \Longrightarrow \quad d q_n + (\alpha_{n+1} q_n + \beta_{n+1} q_{n-1})\sqrt{d} = (\alpha_{n+1} p_n + \beta_{n+1} p_{n-1}) + p_n \sqrt{d}. \end{aligned}$$

Equating coefficients, we obtain

$$d q_n = \alpha_{n+1} p_n + \beta_{n+1} p_{n-1} \quad \text{and} \quad \alpha_{n+1} q_n + \beta_{n+1} q_{n-1} = p_n.$$

Multiplying the first equation by q_n and the second equation by p_n and equating the $\alpha_{n+1} p_n q_n$ terms in each resulting equation, we obtain

$$\begin{aligned} &d q_n^2 - \beta_{n+1} p_{n-1} q_n = p_n^2 - \beta_{n+1} p_n q_{n-1} \\ &\qquad \Longrightarrow \quad p_n^2 - d\, q_n^2 = (p_n q_{n-1} - p_{n-1} q_n) \cdot \beta_{n+1} = (-1)^{n-1} \cdot \beta_{n+1}. \end{aligned}$$

□

Next, we show that *all* solutions of Pell's equation can be found via the convergents of $\sqrt{d}$.

THEOREM 7.36. *Let p_n/q_n denote the n-th convergent of $\sqrt{d}$ and m the period of $\sqrt{d}$. Then the positive integer solutions to*

$$x^2 - d\, y^2 = 1$$

[8] *A person who can, within a year, solve $x^2 - 92y^2 = 1$ is a mathematician. Brahmagupta (598–670).*

are precisely numerators and denominators of the odd convergents of $\sqrt{d}$ *of the form* $x = p_{nm-1}$ *and* $y = q_{nm-1}$*, where* $n > 0$ *is any positive integer for* m *even and* $n > 0$ *is even for* m *odd.*

Proof. We prove our theorem in two steps.

Step 1: We first prove that if $x^2 - d\,y^2 = 1$ with $y > 0$, then x/y is a convergent of $\sqrt{d}$. To see this, observe that since $1 = x^2 - d\,y^2 = (x - \sqrt{d}\,y)(x + \sqrt{d}\,y)$, we have $x - \sqrt{d}\,y = 1/(x + \sqrt{d}\,y)$, so

$$\left|\frac{x}{y} - \sqrt{d}\right| = \left|\frac{x - \sqrt{d}\,y}{y}\right| = \frac{1}{y\,|x + \sqrt{d}\,y|}.$$

Also, $x^2 = d\,y^2 + 1 > d\,y^2$ implies $x > \sqrt{d}\,y$, which implies

$$x + \sqrt{d}\,y > \sqrt{d}\,y + \sqrt{d}\,y = 2\sqrt{d}\,y.$$

Hence,

$$\left|\frac{x}{y} - \sqrt{d}\right| = \frac{1}{y\,|x + \sqrt{d}\,y|} < \frac{1}{y \cdot 2\sqrt{d}\,y} = \frac{1}{2y^2\sqrt{d}} \quad\Longrightarrow\quad \left|\frac{x}{y} - \sqrt{d}\right| < \frac{1}{2y^2}.$$

By Dirichlet's theorem 7.21, it follows that x/y must be a convergent of $\sqrt{d}$.

Step 2: We now finish the proof. By **Step 1** we already know that every solution must be a convergent, so we only need to look for convergents (p_k, q_k) that make $p_k^2 - d\,q_k^2 = 1$. To this end, recall from Lemma 7.35 that

$$p_{k-1}^2 - d\,q_{k-1}^2 = (-1)^k \beta_k,$$

where β_k never equals -1 and k is a multiple of the period of $\sqrt{d}$ if and only if $\beta_k = 1$. In particular, if $p_{k-1}^2 - d\,q_{k-1}^2 = 1$, then as β_k is never equal to -1, we must have $\beta_k = 1$, so k must be a period. Let m be the period of $\sqrt{d}$; then $k = nm$ for some n, in which case

$$p_{nm-1}^2 - d\,q_{nm-1}^2 = (-1)^{nm} \cdot \beta_{nm} = (-1)^{nm} \cdot 1 = (-1)^{nm}.$$

In particular, if m is even, then the right-hand side is one for all n and if m is odd, then the right-hand side is one only for n even. This completes our proof. ☐

The **fundamental solution** of Pell's equation is the smallest positive solution of Pell's equation; here, a solution (x, y) is **positive** means $x, y > 0$. Explicitly, the fundamental solution is (p_{m-1}, q_{m-1}) for an even period m of $\sqrt{d}$ or (p_{2m-1}, p_{2m-1}) for an odd period m.

Example 7.30. Consider the equation $x^2 - 3y^2 = 1$. Since $\sqrt{3} = \langle 1; \overline{1, 2}\rangle$ has period $m = 2$, our theorem says that the positive solutions of $x^2 - 3y^2 = 1$ are precisely $x = p_{2n-1}$ and $y = q_{2n-1}$ for all $n > 0$; that is, $(p_1, q_1), (p_3, q_3), (p_5, q_5), \ldots$. Now the convergents of $\sqrt{3}$ are

n	0	1	2	3	4	5	6	7
$\frac{p_n}{q_n}$	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{5}{3}$	$\frac{7}{4}$	$\frac{19}{11}$	$\frac{26}{15}$	$\frac{71}{41}$	$\frac{97}{56}$

In particular, the fundamental solution is $(2, 1)$ and the rest of the positive solutions are $(7, 4)$, $(26, 15)$, $(97, 56)$, ... Just to verify a couple entries:

$$2^2 - 3 \cdot 1^2 = 4 - 3 = 1$$

and

$$7^2 - 3 \cdot 4^2 = 49 - 3 \cdot 16 = 49 - 48 = 1,$$

and one can continue verifying that all the odd convergents give solutions.

Example 7.31. For another example, consider the equation $x^2 - 13\,y^2 = 1$. In this case, we find that $\sqrt{13} = \langle 3; \overline{1,1,1,1,6} \rangle$ has period $m = 5$. Thus, our theorem says that the positive solutions of $x^2 - 13y^2 = 1$ are precisely $x = p_{5n-1}$ and $y = q_{5n-1}$ for all $n > 0$ *even*; that is, $(p_9, q_9), (p_{19}, q_{19}), (p_{29}, q_{29}), \ldots$. The convergents of $\sqrt{13}$ are

n	0	1	2	3	4	5	6	7	8	9
$\frac{p_n}{q_n}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{7}{2}$	$\frac{11}{3}$	$\frac{18}{5}$	$\frac{119}{33}$	$\frac{137}{38}$	$\frac{256}{71}$	$\frac{393}{109}$	$\frac{649}{180}$

.

In particular, the fundamental solution is $(649, 180)$.

7.9.3. Brahmagupta's algorithm. Thus, to find solutions of Pell's equation we just have to find certain convergents of $\sqrt{d}$. Finding all convergents is quite a daunting task — try finding the solution (p_{19}, q_{19}) for $\sqrt{13}$ — but it turns out that all the positive solutions can be found from the fundamental solution.

Example 7.32. We know that the fundamental solution of $x^2 - 3y^2 = 1$ is $(2, 1)$ and the rest of the positive solutions are $(7, 4)$, $(26, 15)$, $(97, 56)$, ... Observe that

$$(2 + 1 \cdot \sqrt{3})^2 = 4 + 4\sqrt{3} + 3 = 7 + 4\sqrt{3}.$$

Note that the second positive solution $(7, 4)$ to $x^2 - 3y^2 = 1$ appears on the right. Now observe that

$$(2 + 1 \cdot \sqrt{3})^3 = (2 + \sqrt{3})^2\,(2 + \sqrt{3}) = (7 + 4\sqrt{3})\,(2 + \sqrt{3}) = 26 + 15\sqrt{3}.$$

Note that the third positive solution $(26, 15)$ to $x^2 - 3y^2 = 1$ appears on the right. One may conjecture that the n-th positive solution (x_n, y_n) to $x^2 - 3\,y^2 = 1$ is found by multiplying out

$$x_n + y_n\,\sqrt{d} = (2 + 1 \cdot \sqrt{3})^n$$

This is in fact correct as the following theorem shows.

Theorem 7.37 (**Brahmagupta's algorithm**). *If (x_1, y_1) is the fundamental solution of Pell's equation*

$$x^2 - d\,y^2 = 1,$$

then all the other positive solutions (x_n, y_n) can be obtained from the equation

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n \quad , \quad n = 0, 1, 2, 3, \ldots.$$

Proof. To simplify this proof a little, we shall say that $\zeta = x + y\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$ solves Pell's equation to mean that (x, y) solves Pell's equation; similarly, we say ζ is a positive solution to mean that $x, y > 0$. Throughout this proof we shall use the following fact:

$$\zeta \text{ solves Pell's equation} \iff \zeta\,\overline{\zeta} = 1 \ \text{ (that is, } 1/\zeta = \overline{\zeta}). \tag{7.68}$$

This is holds for the simple reason that

$$\xi\,\overline{\xi} = (x + y\sqrt{d})\,(x - y\sqrt{d}) = x^2 - d\,y^2.$$

In particular, if we set $\alpha := x_1 + y_1\sqrt{d}$, then $\alpha\,\overline{\alpha} = 1$ because (x_1, y_1) solves Pell's equation. We now prove our theorem. We first note that (x_n, y_n) is a solution because

$$(x_n + y_n\sqrt{d})\,\overline{(x_n + y_n\sqrt{d})} = \alpha^n \cdot \overline{\alpha^n} = \alpha^n \cdot (\overline{\alpha})^n = (\alpha \cdot \overline{\alpha})^n = 1^n = 1,$$

which in view of (7.68), we conclude that (x_n, y_n) solves Pell's equation. Now suppose that $\xi \in \mathbb{Z}[\sqrt{d}]$ is a positive solution to Pell's equation; we must show that ξ is some power of α. To this end, note that $\alpha \le \xi$ because $\alpha = x_1 + y_1\sqrt{d}$ and (x_1, y_1) is the smallest positive solution of Pell's equation. Since $1 < \alpha$, it follows that $\alpha^k \to \infty$ as $k \to \infty$, so we can choose $n \in \mathbb{N}$ to be the largest natural number such that $\alpha^n \le \xi$. Then, $\alpha^n \le \xi < \alpha^{n+1}$, so dividing by α^n, we obtain

$$1 \le \eta < \alpha \quad \text{where} \quad \eta := \frac{\xi}{\alpha^n} = \xi \cdot (\overline{\alpha})^n,$$

where we used that $1/\alpha = \overline{\alpha}$ from (7.68). Since $\mathbb{Z}[\sqrt{d}]$ is a ring (Lemma 7.30), we know that $\eta = \xi \cdot (\overline{\alpha})^n \in \mathbb{Z}[\sqrt{d}]$ as well. Moreover, η solves Pell's equation because

$$\eta\,\overline{\eta} = \xi \cdot (\overline{\alpha})^n \cdot \overline{\xi} \cdot \alpha^n = (\xi\,\overline{\xi}) \cdot (\overline{\alpha}\,\alpha)^n = 1 \cdot 1 = 1.$$

We shall prove that $\eta = 1$, which shows that $\xi = \alpha^n$. To prove this, observe that from $1 \le \eta < \alpha$ and the fact that $1/\eta = \overline{\eta}$ (since $\eta\,\overline{\eta} = 1$), we have

$$0 < \alpha^{-1} < \eta^{-1} \le 1 \quad \Longrightarrow \quad 0 < \alpha^{-1} < \overline{\eta} \le 1.$$

Let $\eta = p + q\sqrt{d}$ where $p, q \in \mathbb{Z}$. Then the inequalities $1 \le \eta < \alpha$ and $0 < \alpha^{-1} < \overline{\eta} \le 1$ imply that

$$2p = (p + q\sqrt{d}) + (p - q\sqrt{d}) = \eta + \overline{\eta} \ge 1 + \alpha^{-1} > 0$$

and

$$2q\sqrt{d} = (p + q\sqrt{d}) - (p - q\sqrt{d}) = \eta - \overline{\eta} \ge 1 - 1 = 0.$$

In particular, $p > 0$, $q \ge 0$, and $p^2 - dq^2 = 1$ (since η solves Pell's equation). Therefore, $(p, q) = (1, 0)$ or (p, q) is a positive (numerator, denominator) of a convergent of $\sqrt{d}$. However, we know that (x_1, y_1) is the smallest such positive (numerator, denominator), and that $p + q\sqrt{d} = \eta < \alpha = x_1 + y_1\sqrt{d}$. Therefore, we must have $(p, q) = (1, 0)$. This implies that $\eta = 1$ and hence $\xi = \alpha^n$. □

Example 7.33. Since $(649, 180)$ is the fundamental solution to $x^2 - 13\,y^2 = 1$, all the positive solutions are given by

$$x_n + y_n\sqrt{13} = (649 + 180\,\sqrt{13})^n.$$

For instance, for $n = 2$, we find that

$$(649 + 180\,\sqrt{13})^2 = 842401 + 233640\sqrt{13} \quad \Longrightarrow \quad (x_2, y_2) = (842401, 233640),$$

much easier than finding (p_{19}, q_{19}).

There are many cool applications of Pell's equation explored in the exercises. Here's one of my favorites (see Problem 7): Any prime of the form $p = 4k + 1$

is a sum of two squares. This was conjectured by Pierre de Fermat[9] (1601–1665) in 1640 and proved by Euler in 1754. For example, $5, 13, 17$ are such primes, and $5 = 1^2 + 2^2$, $13 = 2^2 + 3^2$, and $17 = 1^2 + 4^2$.

EXERCISES 7.9.

1. Find the fundamental solutions to the equations
$$(a)\ x^2 - 8\,y^2 = 1 \quad , \quad (b)\ x^2 - 5\,y^2 = 1 \quad , \quad (c) x^2 - 7\,y^2 = 1.$$
Using the fundamental solution, find the next two solutions.
2. Here's is a nice problem solvable using continued fractions. A **Pythagorean triple** consists of three natural numbers (x, y, z) such that $x^2 + y^2 = z^2$. For example, $(3, 4, 5)$, $(5, 12, 13)$, and $(8, 15, 17)$ are examples. (Can you find more?) The first example $(3, 4, 5)$ has the property that the first two are consecutive integers; here are some steps to find more Pythagorean triples of this sort.
   (i) Show that (x, y, z) is a Pythagorean triple with $y = x + 1$ if and only if
   $$(2x + 1)^2 - 2z^2 = 1.$$
   (ii) By solving the Pell equation $u^2 - 2\,v^2 = 1$, find the next three Pythagorean triples (x, y, z) (after $(3, 4, 5)$) where x and y are consecutive integers.
3. Here's is another very nice problem that can be solved using continued fractions. Find all triangular numbers that are squares, where recall that a triangular number is of the form $1 + 2 + \cdots + n = n(n + 1)/2$. Here are some steps.
   (i) Show that $n(n + 1)/2 = m^2$ if and only if
   $$(2n + 1)^2 - 8m^2 = 1.$$
   (ii) By solving the Pell equation $x^2 - 8\,y^2 = 1$, find the first three triangular numbers that are squares.
4. The diophantine equation $x^2 - d\,y^2 = -1$ (where $d > 0$ is not a perfect square) is also of interest. In this problem we determine when this equation has solutions. Following the proof of Theorem 7.36, prove the following statements.
   (i) Show that if (x, y) solves $x^2 - d\,y^2 = -1$ with $y > 0$, then x/y is a convergent of $\sqrt{d}$.
   (ii) Prove that $x^2 - d\,y^2 = -1$ has a solution if and only if the period of $\sqrt{d}$ is odd, in which case the nonnegative solutions are exactly $x = p_{nm-1}$ and $y = q_{nm-1}$ for all $n > 0$ odd.
5. Which of the following equations have solutions? If an equation has solutions, find the fundamental solution.
$$(a)\ x^2 - 2\,y^2 = -1 \quad , \quad (b)\ x^2 - 3\,y^2 = -1 \quad , \quad (c) x^2 - 17\,y^2 = -1.$$
6. In this problem we prove that the diophantine equation $x^2 - p\,y^2 = -1$ always has a solution if p is a prime number of the form $p = 4k + 1$ for an integer k. For instance, since $13 = 4 \cdot 3 + 1$ and $17 = 4 \cdot 4 + 1$, $x^2 - 13y^2 = -1$ and $x^2 - 17y^2 = -1$ have solutions (as you already saw in the previous problem). Let $p = 4k + 1$ be prime.
   (i) Let (x_1, y_1) be the fundamental solution of $x^2 - p\,y^2 = 1$. Prove that x_1 and y_1 cannot both be even and cannot both be odd.
   (ii) Show that the case x_1 is even and y_1 is odd cannot happen. Suggestion: Write $x_1 = 2a$ and $y_1 = 2b + 1$ and plug this into $x_1^2 - p\,y_1^2 = 1$.

[9] *[In the margin of his copy of Diophantus' Arithmetica, Fermat wrote] To divide a cube into two other cubes, a fourth power or in general any power whatever into two powers of the same denomination above the second is impossible, and I have assuredly found an admirable proof of this, but the margin is too narrow to contain it. Pierre de Fermat (1601–1665). Fermat's claim in this marginal note, later to be called "Fermat's last theorem" remained an unsolved problem in mathematics until 1995 when Andrew Wiles (1953 –) finally proved it.*

(iii) Thus, we may write $x_1 = 2a+1$ and $y_1 = 2b$. Show that $p\,b^2 = a\,(a+1)$. Conclude that p must divide a or $a+1$.
(iv) Suppose that p divides a; that is, $a = mp$ for an integer m. Show that $b^2 = m\,(mp+1)$ and that m and $mp+1$ are relatively prime. Using this equality, prove that $m = s^2$ and $mp+1 = t^2$ for integers s, t. Conclude that $t^2 - p\,s^2 = 1$ and derive a contradiction.
(v) Thus, it must be the case that p divides $a+1$. Using this fact and an argument similar to the one in the previous step, find a solution to $x^2 - d\,y^2 = -1$.

7. In this problem we prove the following incredible result of Euler: Every prime of the form $p = 4k+1$ can be expressed as the sum of two squares.
(i) Let $p = 4k+1$ be prime. Using the previous problem and Problem 4, prove that the period of $\sqrt{p}$ is odd and from this, deduce that $\sqrt{p}$ has the expansion

$$\sqrt{p} = \langle a_0; \overline{a_1, a_2, \ldots, a_{\ell-1}, a_\ell, a_\ell, a_{\ell-1}, \ldots, a_1, 2a_0}\rangle.$$

(ii) Let η be the complete quotient $\xi_{\ell+1}$:

$$\eta := \xi_{\ell+1} = \langle \overline{a_\ell, a_{\ell-1} \ldots, a_1, 2a_0, a_1, \ldots, a_{\ell-1}, a_\ell}\rangle.$$

Prove that $-1 = \eta \cdot \overline{\eta}$. Suggestion: Use Lemma 7.33.
(iii) Finally, writing $\eta = (a+\sqrt{p})/b$ (why does η have this form?) show that $p = a^2+b^2$.

7.10. Epilogue: Transcendental numbers, π, e, and where's calculus?

It's time to get a tissue box, because, unfortunately, our adventures through Book I have come to an end. However, in Book II we start a new amazing journey through topology and calculus. In this section we wrap up Book I with a discussion on transcendental numbers and continued fractions.

7.10.1. Approximable numbers. A real number ξ is said to be **approximable** (by rationals) to order $n \geq 1$ if there exists a constant C and infinitely many rational numbers p/q in lowest terms with $q > 0$ such that

$$\left|\xi - \frac{p}{q}\right| < \frac{C}{q^n}. \tag{7.69}$$

Observe that if ξ is approximable to order $n > 1$, then it is automatically approximable to $n-1$; this is because

$$\left|\xi - \frac{p}{q}\right| < \frac{C}{q^n} \leq \frac{C}{q^{n-1}}.$$

Similarly, if ξ approximable to any order k with $1 \leq k \leq n$. Intuitively, the approximability order n measures how close we can surround ξ with "good" rational numbers, that is, rational numbers having small denominators. To see what this means, suppose that ξ is only approximable to order 1. Thus, there is a C and infinitely many rational numbers p/q in lowest terms with $q > 0$ such that

$$\left|\xi - \frac{p}{q}\right| < \frac{C}{q}.$$

This inequality suggests that in order to find rational numbers very close to ξ, these rational numbers need to have large denominators to make C/q small. However, if ξ were approximable to order 1000, then there is a C and infinitely many rational numbers p/q in lowest terms with $q > 0$ such that

$$\left|\xi - \frac{p}{q}\right| < \frac{C}{q^{1000}}.$$

This inequality suggests that in order to find rational numbers very close to ξ, these rational numbers don't need to have large denominators, because even for small q, the large power of 1000 will make C/q^{1000} small. The following lemma shows that there is a limit to how close we can surround algebraic numbers by "good" rational numbers.

LEMMA 7.38. *If ξ is real algebraic of degree $n \geq 1$ (so ξ is rational if $n = 1$), then there exists a constant $c > 0$ such that for all rational numbers $p/q \neq \xi$ with $q > 0$, we have*

$$\left|\xi - \frac{p}{q}\right| \geq \frac{c}{q^n}.$$

PROOF. Assume that $f(\xi) = 0$ where

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0, \qquad a_k \in \mathbb{Z},$$

and that no such polynomial function of lower degree has this property. First, we claim that $f(r) \neq 0$ for any rational number $r \neq \xi$. Indeed, if $f(r) = 0$ for some rational number $r \neq \xi$, then we can write $f(x) = (x-r)g(x)$ where g is a polynomial of degree $n-1$. Then $0 = f(\xi) = (\xi - r)g(\xi)$ implies, since $\xi \neq r$, that $g(\xi) = 0$. This implies that the degree of ξ is $n-1$ contradicting the fact that the degree of ξ is n. Now for any rational $p/q \neq \xi$ with $q > 0$, we see that

$$\begin{aligned} 0 \neq |f(p/q)| &= \left|a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_1\left(\frac{p}{q}\right) + a_0\right| \\ &= \frac{|a_n p^n + a_{n-1}p^{n-1}q + \cdots + a_1 p q^{n-1} + a_0 q^n|}{q^n}. \end{aligned}$$

The numerator is a nonnegative integer, which cannot be zero, so the numerator must be ≥ 1. Therefore,

$$|f(p/q)| \geq 1/q^n \quad \text{for all rational numbers } p/q \neq \xi \text{ with } q > 0. \tag{7.70}$$

Second, we claim that there is an $M > 0$ such that

$$|x - \xi| \leq 1 \quad \Longrightarrow \quad |f(x)| \leq M|x - \xi|. \tag{7.71}$$

Indeed, note that since $f(\xi) = 0$, we have

$$f(x) = f(x) - f(\xi) = a_n(x^n - \xi^n) + a_{n-1}(x^{n-1} - \xi^{n-1}) + \cdots + a_1(x - \xi).$$

Since

$$x^k - \xi^k = (x - \xi)\, q_k(x), \quad q_k(z) = x^{k-1} + x^{k-2}\,\xi + \cdots + x\,\xi^{k-2} + \xi^{k-1},$$

plugging each of these, for $k = 1, 2, 3, \ldots, n$, into the previous equation for $f(x)$, we see that $f(x) = (x - \xi)h(x)$ where h is a continuous function. In particular, since $[\xi - 1, \xi + 1]$ is a closed and bounded interval, there is an M such that $|h(x)| \leq M$ for all $x \in [\xi - 1, \xi + 1]$. This proves our claim.

Finally, let $p/q \neq \xi$ be a rational number with $q > 0$. If $|\xi - p/q| > 1$, then

$$\left|\xi - \frac{p}{q}\right| > 1 \geq \frac{1}{q^n}.$$

If $|\xi - p/q| \leq 1$, then by (7.70) and (7.71), we have

$$\left|\xi - \frac{p}{q}\right| \geq \frac{1}{M}|f(p/q)| \geq \frac{1}{M}\frac{1}{q^n}.$$

Hence, $|\xi - p/q| \geq c/q^n$ for all rational $p/q \neq \xi$ with $q > 0$, where c is the smaller of 1 and $1/M$. □

Let us negate the statement of this lemma: If for all constants $c > 0$, there exists a rational number $p/q \neq \xi$ with $q > 0$ such that

$$\left|\xi - \frac{p}{q}\right| < \frac{c}{q^n}, \tag{7.72}$$

then ξ is not algebraic of degree $n \geq 1$. Since a transcendental number is a number that is not algebraic of any degree n, we can think of a transcendental number as a number that can be surrounded arbitrarily close by "good" rational numbers. This leads us to Liouville numbers to be discussed shortly, but before talking about these special transcendental numbers, we use our lemma to prove the following important result.

THEOREM 7.39. *A real algebraic number of degree n is not approximable to order $n+1$ (and hence not to any higher order). Moreover, a rational number is approximable to order* 1 *and a real number is irrational if and only if it is approximable to order* 2.

PROOF. Let ξ be algebraic of degree $n \geq 1$ (so ξ is rational if $n = 1$). Then by Lemma 7.38, there exists a constant c such that for all rational numbers $p/q \neq \xi$ with $q > 0$, we have

$$\left|\xi - \frac{p}{q}\right| \geq \frac{c}{q^n}.$$

It follows that ξ is not approximable by rationals to order $n+1$ because

$$\left|\xi - \frac{p}{q}\right| < \frac{C}{q^{n+1}} \quad \Longrightarrow \quad \frac{c}{q^n} < \frac{C}{q^{n+1}} \quad \Longrightarrow \quad q < C/c.$$

Since there are only finitely many integers q such that $q < C/c$; it follows that there are only finitely many fractions p/q such that $|\xi - p/q| < C/q^{n+1}$.

Let a/b be a rational number in lowest terms with $b \geq 1$; we shall prove that a/b is approximable to order 1. (Note that we already know from our first statement that a/b is not approximable to order 2.) From Theorem 7.9, we know that the equation $ax - by = 1$ has an infinite number of integer solutions (x, y). The solutions (x, y) are automatically relatively prime. Moreover, if (x_0, y_0) is any one integral solution, then all solutions are of the form

$$x = x_0 + bt \quad , \quad y = y_0 + at \quad , \quad t \in \mathbb{Z}.$$

Since $b \geq 1$ we can choose t large so as to get infinitely many solutions with $x > 0$. With $x > 0$, we see that

$$\left|\frac{a}{b} - \frac{y}{x}\right| = \left|\frac{ax - by}{bx}\right| = \frac{1}{bx} < \frac{2}{x},$$

which shows that a/b is approximable to order 1.

Finally, if a number is irrational, then it is approximable to order 2 from Dirichlet's approximation theorem 7.21; conversely, if a number is approximable to order 2, then it must be irrational by the first statement of this theorem. $\square$

Using this theorem we can prove that certain numbers must be irrational. For instance, let $\{a_n\}$ be any sequence of $0, 1$'s where there are infinitely many 1's. consider

$$\xi = \sum_{n=0}^{\infty} \frac{a_n}{2^{2^n}}.$$

Note that ξ is the real number with binary expansion $a_0.0a_10a_30\cdots$, with a_n in the 2^n-th decimal place and with zeros everywhere else. Any case, fix a natural number n with $a_n \neq 0$ and let $s_n = \sum_{k=0}^{n} \frac{a_k}{2^{2^k}}$ be the n-th partial sum of this series. Then we can write s_n as p/q where $q = 2^{2^n}$. Observe that

$$\begin{aligned}
\left|\xi - s_n\right| &\leq \frac{1}{2^{2^{n+1}}} + \frac{1}{2^{2^{n+2}}} + \frac{1}{2^{2^{n+3}}} + \frac{1}{2^{2^{n+4}}} + \cdots \\
&< \frac{1}{2^{2^{n+1}}} + \frac{1}{2^{2^{n+1}+1}} + \frac{1}{2^{2^{n+1}+2}} + \frac{1}{2^{2^{n+1}+3}} + \cdots \\
&= \frac{1}{2^{2^{n+1}}}\left(1 + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right) = \frac{2}{2^{2^{n+1}}} = \frac{2}{(2^{2^n})^2}.
\end{aligned}$$

In conclusion,

$$\left|\xi - s_n\right| < \frac{2}{(2^{2^n})^2} = \frac{C}{q^2},$$

where $C = 2$. Thus, ξ is approximable to order 2, and hence must be irrational.

7.10.2. Liouville numbers. Numbers that satisfy (7.72) with $c = 1$ are special: A real number ξ is called a **Liouville number**, after Joseph Liouville (1809–1882), if for every natural number n there is a rational number $p/q \neq \xi$ with $q > 1$ such that

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^n}.$$

These numbers are transcendental by our discussion around (7.72). Because this fact is so important, we state this as a theorem.

THEOREM 7.40 (**Liouville's theorem**). *Any Liouville number is transcendental.*

Using Liouville's theorem we can give many (in fact uncountably many — see Problem 3) examples of transcendental numbers. Let $\{a_n\}$ be any sequence of integers in $0, 1, \ldots, 9$ where there are infinitely many nonzero integers. Let

$$\xi = \sum_{n=0}^{\infty} \frac{a_n}{10^{n!}}.$$

Note that ξ is the real number with decimal expansion

$$a_0.a_1a_2000a_3000000000000000000a_4\cdots,$$

with a_n in the $n!$-th decimal place and with zeros everywhere else. Using Liouville's theorem we'll show that ξ is transcendental. Fix a natural number n with $a_n \neq 0$ and let s_n be the n-th partial sum of this series. Then s_n can be written as p/q where $q = 10^{n!} > 1$. Observe that

$$\begin{aligned}
\left|\xi - s_n\right| &\leq \frac{9}{10^{(n+1)!}} + \frac{9}{10^{(n+2)!}} + \frac{9}{10^{(n+3)!}} + \frac{9}{10^{(n+4)!}} + \cdots \\
&< \frac{9}{10^{(n+1)!}} + \frac{9}{10^{(n+1)!+1}} + \frac{9}{10^{(n+1)!+2}} + \frac{9}{10^{(n+1)!+3}} + \cdots \\
&= \frac{9}{10^{(n+1)!}}\left(1 + \frac{1}{10^1} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots\right) \\
&= \frac{10}{10^{(n+1)!}} = \frac{10}{10^{n\cdot n!} \cdot 10^{n!}} \leq \frac{1}{10^{n\cdot n!}}.
\end{aligned}$$

In conclusion,

$$\left|\xi - s_n\right| < \frac{1}{(10^{n!})^n} = \frac{1}{q^n},$$

so ξ is a Liouville number and therefore is transcendental.

7.10.3. Continued fractions and the "most extreme" irrational of all irrational numbers. We now show how continued fractions can be used to *construct* transcendental numbers! This is achieved by the following simple observation. Let $\xi = \langle a_0; a_1, \ldots\rangle$ be an irrational real number with convergents $\{p_n/q_n\}$. Then by our fundamental approximation theorem 7.18, we know that

$$\left|\xi - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}.$$

Since

$$q_n q_{n+1} = q_n(a_{n+1}q_n + q_{n-1}) \geq a_{n+1}q_n^2,$$

we see that

$$\left|\xi - \frac{p_n}{q_n}\right| < \frac{1}{a_{n+1}\, q_n^2}. \tag{7.73}$$

Thus, we can make the rational number p_n/q_n approximate ξ as close as we wish by simply taking the next partial quotient a_{n+1} larger. We use this observation in the following theorem.

THEOREM 7.41. *Let $\varphi : \mathbb{N} \to (0, \infty)$ be a function. Then there is an irrational number ξ and infinitely many rational numbers p/q such that*

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{\varphi(q)}.$$

PROOF. We define $\xi = \langle a_0; a_1, a_2, \ldots\rangle$ by choosing the a_n's inductively as follows. Let $a_0 \in \mathbb{N}$ be arbitrary. Assume that $a_0, \ldots, a_n$ have been chosen. With q_n the denominator of $\langle a_0; a_1, \ldots, a_n\rangle$, choose (via Archimedean) $a_{n+1} \in \mathbb{N}$ such that

$$a_{n+1}q_n^2 > \varphi(q_n).$$

This defines the a_n's. Now defining $\xi := \langle a_0; a_1, a_2, \ldots\rangle$, by (7.73), for any natural number n we have

$$\left|\xi - \frac{p_n}{q_n}\right| < \frac{1}{a_{n+1}\, q_n^2} < \frac{1}{\varphi(q_n)}.$$

This completes our proof. ☐

Using this theorem we can easily find transcendental numbers. For example, with $\varphi(q) = e^q$, we can find an irrational ξ such that for infinitely many rational numbers p/q, we have

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{e^q}.$$

Since for any $n \in \mathbb{N}$, we have $e^q = \sum_{k=0}^{\infty} q^k/k! > q^n/n!$, it follows that for infinitely many rational numbers p/q, we have

$$\left|\xi - \frac{p}{q}\right| < \frac{c}{q^n},$$

where $c = n!$. In particular, ξ is transcendental.

As we have just seen, we can form transcendental numbers by choosing the partial quotients in an infinite simple continued fraction to be very large and transcendental numbers are the irrational numbers which are "closest" to good rational numbers. With this in mind, we can think of infinite continued fractions with small partial quotients as far from being transcendental or far from rational. Since 1 is the smallest natural number, we can consider the golden ratio

$$\Phi = \frac{1+\sqrt{5}}{2} = \langle 1; 1, 1, 1, 1, 1, 1, 1, \ldots \rangle$$

as being the "most extreme" or "most irrational" of all irrational numbers in the sense that it is the "farthest" irrational number from being transcendental or the "farthest" irrational number from being rational.

7.10.4. What about π and e and what about calculus? Above we have already seen examples (in fact, uncountably many — see Problem 3) of transcendental numbers and we even know how to construct them using continued fractions. However, these numbers seem in some sense to be "artificially" made. What about numbers that are more "natural" such as π and e? Are these numbers transcendental? In fact, these numbers do turn out to be transcendental, but the "easiest" proofs of these facts need the technology of calculus (derivatives)! Our next adventure is to study calculus and during our journey we'll prove that π and e are transcendental. However, before going on this adventure, we ask you to look back at all the amazing things that we've encountered during these past chapters — everything without using one single derivative or integral!

Exercises 7.10.

1. Given any integer $b \geq 2$, prove that $\xi = \sum_{n=0}^{\infty} b^{-2^n}$ is irrational.
2. Let $b \geq 2$ be an integer and let $\{a_n\}$ be any sequence of integers $0, 1, \ldots, b-1$ where there are infinitely many nonzero a_n's. Prove that $\xi = \sum_{n=1}^{\infty} a_n b^{-n!}$ is transcendental.
3. Using a Cantor diagonal argument as in the proof of Theorem 3.35, prove that the set of all numbers of the form $\xi = \sum_{n=0}^{\infty} \frac{a_n}{10^{n!}}$ where $a_n \in \{0, 1, 2, \ldots, 9\}$ is uncountable. That is, assume that the set of all such numbers is countable and construct a number of the same sort not in the set. Since we already showed that all these numbers are Liouville numbers, they are transcendental, so this argument provides another proof that the set of all transcendental numbers is uncountable.
4. Going through the construction of Theorem 7.41, define $\xi \in \mathbb{R}$ such that if $\{p_n/q_n\}$ are the convergents of its canonical continued fraction expansion, then for all n,
$$\left| \xi - \frac{p_n}{q_n} \right| < \frac{1}{q_n^n}.$$
Show that ξ is a Liouville number, and hence is transcendental.