SOLUTIONS TO SELECTED PUTNAM PROBLEMS: 96 A-1, 93 B-1

Problem. (1996 A-1) Find the least number A such that for any two squares of combined area 1, a rectangle of area A exists such that the two squares can be packed into the rectange (without interior overlap). You may assume that the sides of the squares are parallel to the sides of the rectangle.

Solution of Problem A-1 (1996). Let x be the side length of the smaller square, if there is one. The other square must have side length $\sqrt{1-x^2}$. If $x = \frac{\sqrt{2}}{2}$, then both squares have the same size, so we need only consider values of x between 0 and $\frac{\sqrt{2}}{2}$.

The smallest rectangle containing both squares has the length of one side is equal to the side length of the larger square, and the length of the other side equal to the sum of the side lengths of the two squares. The area of this rectangle is $A(x) = (\sqrt{1-x^2}+x)\sqrt{1-x^2} = 1 - x^2 + x\sqrt{1-x^2}$. The problem asks for the maximum value of $A(x)$.

We compute

$$A'(x) = -2x + \sqrt{1-x^2} + x(\frac{1}{2})(1-x^2)^{-\frac{1}{2}} \cdot (-2x) = -2x + \frac{1-2x^2}{\sqrt{1-x^2}}.$$

Requiring $A'(x) = 0$ is, after some algebra, the same as requiring $8x^4 - 8x^2 + 1 = 0$; so $x^2 = \frac{1}{2} \pm \frac{\sqrt{2}}{4}$. Since $x \leq \sqrt{2}/2$, $x^2 \leq \frac{1}{2}$, and we may take the smaller root. Thus, $x^2 = \frac{1}{2} - \frac{\sqrt{2}}{4}$.

For this value of x,

$$A(x) = 1 - x^2 + x\sqrt{1-x^2} = 1 - (\frac{1}{2} - \frac{\sqrt{2}}{4}) + \sqrt{\frac{1}{2} - \frac{\sqrt{2}}{4}} \cdot (\sqrt{1 - (\frac{1}{2} - \frac{\sqrt{2}}{4})}) = \frac{1+\sqrt{2}}{2}.$$

At the endpoints, we have $A(0) = 1$, and $A(\sqrt{2}/2) = 1$, so the maximum is $A = \frac{1+\sqrt{2}}{2}$. $\square$

Problem. Find the smallest positive integer n such that for every integer m with $0 < m < 1993$, there exists an integer k for which

$$\frac{m}{1993} < \frac{k}{n} < \frac{m+1}{1994}.$$

Solution of Problem 1993 B-1. The smallest n is $3987 = 1993 + 1994$.

First we note that this n actually works. To see this, we require the following

Lemma 1. *Let a, b, c, d be positive integers with $\frac{a}{b} < \frac{c}{d}$. Then*

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$

Proof. Cross-multiply to see that the inequality of the hypothesis is equivalent to $ad < bc$.

Add ab to both sides so we have $ab + ad < ab + bc$, or $a(b+d) < b(a+c)$. This implies the first inequality of the conclusion.

Adding cd to both sides gives $d(a+c) < c(b+d)$, which is equivalent to the second inequality of the conclusion.

This completes the proof of the lemma.

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From the lemma, it is easy to see that $n = 1993 + 1994$ works. Take $k = 2m + 1$, and from the lemma we have

$$\frac{m}{1993} < \frac{k}{n} < \frac{m+1}{1994}$$

for each m with $0 < m < 1993$.

Now we show that no smaller n will do. To see this, take $m = 1992$, so that there must exist a k with

$$\frac{1992}{1993} < \frac{k}{n} < \frac{1993}{1994}.$$

For this k we have

$$1 - \frac{1992}{1993} > 1 - \frac{k}{n} > 1 - \frac{1993}{1994},$$

or

$$\frac{1}{1993} > \frac{n-k}{n} > \frac{1}{1994}.$$

Inverting everything gives:

$$1993 < \frac{n}{n-k} < 1994,$$

or

$$1993(n-k) < n < 1994(n-k).$$

Since n is an integer, we must have $n-k \geq 2$. If $n-k = 2$, then the only n that works is $2 \cdot 1993 + 1 = 3987$. For larger values of $n-k$, larger ns are required, so 3987 is the least possible n. $\square$

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