MATH 488A: PUTNAM PROBLEMS LEVEL 1

1. 1985–1989

Problem. (1985 A-1) Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that

(i) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and
(ii) $A_1 \cap A_2 \cap A_3 = \emptyset$.

Express your answer in the form $2^a 3^b 5^c 7^d$, where a, b, c, d are nonnegative integers.

Problem. (1985 B-1) Let k be the smallest positive integer for which there exist distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial

$$p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$

has exactly k nonzero coefficients. Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved.

Problem. (1986 A-1) Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \le 13x^2$.

Problem. (1986 B-1) Inscribe a rectangle of base b and height h in a circle of radius one, and inscribe an isosceles triangle in the region of the circle cut off by one base of the rectangle (with that side as the base of the triangle). For what value of h do the rectangle and triangle have the same area?

Problem. (1987 A-1) Curves A, B, C and D are defined in the plane as follows:

$$\begin{aligned}
A &= \left\{(x, y) : x^2 - y^2 = \frac{x}{x^2 + y^2}\right\}, \\
B &= \left\{(x, y) : 2xy + \frac{y}{x^2 + y^2} = 3\right\}, \\
C &= \left\{(x, y) : x^3 - 3xy^2 + 3y = 1\right\}, \\
D &= \left\{(x, y) : 3x^2 y - 3x - y^3 = 0\right\}.
\end{aligned}$$

Prove that $A \cap B = C \cap D$.

Problem. (1987 B-1) Evaluate

$$\int_2^4 \frac{\sqrt{\ln(9 - x)}\, dx}{\sqrt{\ln(9 - x)} + \sqrt{\ln(x + 3)}}.$$

Problem. (1988 A-1) Let R be the region consisting of the points (x, y) of the cartesian plane satisfying both $|x| - |y| \le 1$ and $|y| \le 1$. Sketch the region R and find its area.

Problem. (1988 B-1) A *composite* (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \dots\}$. Show that every composite is expressible as $xy + xz + yz + 1$, with x, y, z positive integers.

Date: November 11, 2002.

Problem. (1989 A-1) How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1?

Problem. (1989 B-1) A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $\frac{a\sqrt{b}+c}{d}$, where a, b, c, d are integers.

2. The 90s

Problem. (1990 A-1)

$$T_0 = 2, T_1 = 3, T_2 = 6,$$

and for $n \geq 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$

The first few terms are

$$2, 3, 6, 14, 40, 152, 784, 5168, 40576.$$

Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where $\{A_n\}$ and $\{B_n\}$ are well-known sequences.

Problem. (1990 B-1) Find all real-valued continuously differentiable functions f on the real line such that for all x,

$$(f(x))^2 = \int_0^x [(f(t))^2 + (f'(t))^2]\, dt + 1990.$$

Problem. (1991 A-1) A 2×3 rectangle has vertices as $(0,0), (2,0), (0,3)$, and $(2,3)$. It rotates $90°$ clockwise about the point $(2,0)$. It then rotates $90°$ clockwise about the point $(5,0)$, then $90°$ clockwise about the point $(7,0)$, and finally, $90°$ clockwise about the point $(10,0)$. (The side originally on the x-axis is now back on the x-axis.) Find the area of the region above the x-axis and below the curve traced out by the point whose initial position is (1,1).

Problem. (1991 B-1) For each integer $n \geq 0$, let $S(n) = n - m^2$, where m is the greatest integer with $m^2 \leq n$. Define a sequence $(a_k)_{k=0}^{\infty}$ by $a_0 = A$ and $a_{k+1} = a_k + S(a_k)$ for $k \geq 0$. For what positive integers A is this sequence eventually constant?

Problem. (1992 A-1) Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be a functions that satisfies the following conditions.

(i) $f(f(n)) = n$, for all integers n;
(ii) $f(f(n+2)+2) = n$ for all integers n;
(iii) $f(0) = 1$.

Show that $f(n) = 1 - n$ for all $n \in \mathbb{Z}$.

Problem. (1992 B-1) Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S. For a given $n \geq 2$, what is the smallest possible number of elements in A_S?

Problem. (1993 A-1) The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find c so that the areas of the two shaded regions are equal. [Figure not included. The first region is bounded by the y-axis, the line $y = c$ and the curve; the other lies under the curve and above the line $y = c$ between their two points of intersection.]

Problem. (1993 B-1) Find the smallest positive integer n such that for every integer m with $0 < m < 1993$, there exists an integer k for which

$$\frac{m}{1993} < \frac{k}{n} < \frac{m+1}{1994}.$$

Problem. (1994 A-1) Let (a_n) be a sequence of positive reals such that, for all n, $a_n \leq a_{2n} + a_{2n+1}$. Prove that $\sum_{n=1}^{\infty} a_n$ diverges.

Problem. (1994 B-1) Find all positive integers n such that $|n - m^2| \leq 250$ for exactly 15 nonnegative integers m.

Problem. (1995 A-1) Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any *three* (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.

Problem. (1995 B-1) For a partition π of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. [A *partition* of a set S is a collection of disjoint subsets (parts) whose union is S.]

Problem. (1996 A-1) Find the least number A such that for any two squares of combined area 1, a rectangle of area A exists such that the two squares can be packed in the rectangle (without interior overlap). You may assume that the sides of the squares are parallel to the sides of the rectangle.

Problem. (1996 B-1) Define a **selfish** set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish.

Problem. (1997 A-1) A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle ABC has H as the intersection of the altitudes, O the center of the circumscribed circle, M the midpoint of BC, and F the foot of the altitude from A. What is the length of BC?

Problem. (1997 B-1) Let $\{x\}$ denote the distance between the real number x and the nearest integer. For each positive integer n, evaluate

$$F_n = \sum_{m=1}^{6n-1} \min(\{\frac{m}{6n}\}, \{\frac{m}{3n}\}).$$

(Here $\min(a, b)$ denotes the minimum of a and b.)

Problem. (1998 A-1) A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

Problem. (1998 B-1) Find the minimum value of

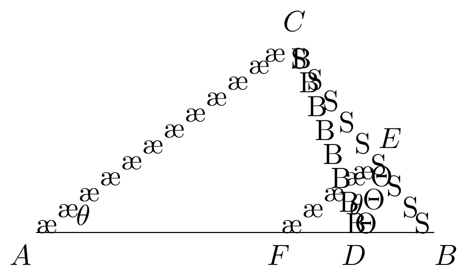
$$\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}$$

for $x > 0$.

Problem. (1999 A-1) Find polynomials $f(x)$, $g(x)$ and $h(x)$, if they exist, such that, for all x,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & x < -1 \\ 3x+2 & -1 \le x \le 0 \\ -2x+2 & x > 0. \end{cases}$$

Problem. (1999 B-1) Right triangle ABC has right angle at C and $\angle BAC = \theta$; the point D is chosen on AB so that $|AC| = |AD| = 1$; the point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F. Evaluate $\lim_{\theta \to 0} |EF|$. [Here, $|PQ|$ denotes the length of the line segment PQ.]



3. The 00s

Problem. (2000 A-1) Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, x_2, \ldots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?

Problem. (2000 B-1) Let a_j, b_j, and c_j be integers for $1 \le j \le N$. Assume, for each j, that at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j, $1 \le j \le N$.

Problem. (2001 A-1) Consider a set S and a binary operation $*$ on S (that is, for each $a, b \in S$, $a * b$ is in S). Assume that $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.

Problem. (2001 B-1) Let n be an even positive integer. Write the numbers $1, 2, ..., n^2$ in the squares of an $n \times n$ grid so that the k-th row, from left to right, is

$$(k-1)n + 1, (k-1)n + 2, ..., (k-1)n + n.$$

Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each such coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

Problem. (2002 A-1) Let k be a positive integer. The n-th derivative of $1/(x^k - 1)$ has the form $P_n(x)/(x^k - 1)^{n+1}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

Problem. (2002 B-1) Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?

Problem. (2003 A-1) Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with k an arbitrary positive integer and $a_1 \le a_2 \le \cdots \le a_k \le a_1 + 1$? For example, with $n = 4$ there are four ways: $4, 2+2, 1+1+2, 1+1+1+1$.

Problem. (2003 B-1) Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

Problem. (2004 A-1) Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N ?

Problem. (2004 B-1) Let $P(x) = c_n x^n + c_{n-1}x^{n-1} + ... + c_0$ be a polynomial with integer coefficients. Suppose that r is a rational number such that $P(r) = 0$. Show that the n numbers

$$c_n r, c_n r^2 + c_{n-1}, c_n r^3 + c_{n-1}r^2 + c_{n-2}r, ..., c_n r^n + c_{n-1}r^{n-1} + ... + c_1 r$$

are integers.

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