MATH 488A: PUTNAM PROBLEMS LEVEL 2

1. 1985–1989

Problem. (1985 A-2) Let T be an acute triangle. Inscribe a rectangle R in T with one side along a side of T. Then inscribe a rectangle S in the triangle formed by the side of R opposite the side on the boundary of T, and the other two sides of T, with one side along the side of R. For any polygon X, let $A(X)$ denote the area of X. Find the maximum value, or show that no maximum exists, of $\frac{A(R)+A(S)}{A(T)}$, where T ranges over all triangles and R, S over all rectangles as above.

Problem. (1985 B-2) Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$, and

$$\frac{d}{dx} f_{n+1}(x) = (n+1) f_n(x+1)$$

for $n \geq 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

Problem. (1986 A-2) What is the units (i.e., rightmost) digit of

$$\left\lfloor \frac{10^{20000}}{10^{100}+3} \right\rfloor ?$$

Here $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Problem. (1986 B-2) Prove that there are only a finite number of possibilities for the ordered triple $T = (x-y, y-z, z-x)$, where x, y, z are complex numbers satisfying the simultaneous equations

$$x(x-1) + 2yz = y(y-1) + 2zx + z(z-1) + 2xy,$$

and list all such triples T.

Problem. (1987 A-2) The sequence of digits

$$123456789101112131415161718192021\ldots$$

is obtained by writing the positive integers in order. If the 10^n-th digit in this sequence occurs in the part of the sequence in which the m-digit numbers are placed, define $f(n)$ to be m. For example, $f(2) = 2$ because the 100th digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(1987)$.

Problem. (1987 B-2) Let r, s and t be integers with $0 \leq r$, $0 \leq s$ and $r + s \leq t$. Prove that

$$\frac{\binom{s}{0}}{\binom{t}{r}} + \frac{\binom{s}{1}}{\binom{t}{r+1}} + \cdots + \frac{\binom{s}{s}}{\binom{t}{r+s}} = \frac{t+1}{(t+1-s)\binom{t-s}{r}}.$$

Problem. (1988 A-2) A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b).

Problem. (1988 B-2) Prove or disprove: If x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.

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Problem. (1989 A-2) Evaluate $\int_0^a \int_0^b e^{\max\{b^2x^2,a^2y^2\}}\,dy\,dx$ where a and b are positive.

Problem. (1989 B-2) Let S be a non-empty set with an associative operation that is left and right cancellative ($xy = xz$ implies $y = z$, and $yx = zx$ implies $y = z$). Assume that for every a in S the set $\{a^n : n = 1, 2, 3, \ldots\}$ is finite. Must S be a group?

2. The 90s

Problem. (1990 A-2) Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$ $(n, m = 0, 1, 2, \ldots)$?

Problem. (1990 B-2) Prove that for $|x| < 1$, $|z| > 1$,

$$1 + \sum_{j=1}^{\infty} (1 + x^j) P_j = 0,$$

where P_j is

$$\frac{(1-z)(1-zx)(1-zx^2)\cdots(1-zx^{j-1})}{(z-x)(z-x^2)(z-x^3)\cdots(z-x^j)}.$$

Problem. (1991 A-2) Let A and B be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2B = B^2A$, can $A^2 + B^2$ be invertible?

Problem. (1991 B-2) Suppose f and g are non-constant, differentiable, real-valued functions defined on $(-\infty, \infty)$. Furthermore, suppose that for each pair of real numbers x and y,

$$\begin{aligned} f(x+y) &= f(x)f(y) - g(x)g(y), \\ g(x+y) &= f(x)g(y) + g(x)f(y). \end{aligned}$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.

Problem. (1992 A-2) Define $C(\alpha)$ to be the coefficient of x^{1992} in the power series about $x = 0$ of $(1 + x)^\alpha$. Evaluate

$$\int_0^1 \left(C(-y-1) \sum_{k=1}^{1992} \frac{1}{y+k} \right) dy.$$

Problem. (1992 B-2) For nonnegative integers n and k, define $Q(n, k)$ to be the coefficient of x^k in the expansion of $(1 + x + x^2 + x^3)^n$. Prove that

$$Q(n,k) = \sum_{j=0}^{k} \binom{n}{j} \binom{n}{k-2j},$$

where $\binom{a}{b}$ is the standard binomial coefficient. (Reminder: For integers a and b with $a \geq 0$, $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ for $0 \leq b \leq a$, with $\binom{a}{b} = 0$ otherwise.)

Problem. (1993 A-2) Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$. Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

Problem. (1993 B-2) Consider the following game played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of two players. Beginning with A, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n + 1$. The last person to discard wins the game. Assuming optimal strategy by both A and B, what is the probability that A wins?

Problem. (1994 A-2) Find the positive value of m such that the area in the first quadrant enclosed by the ellipse $\frac{x^2}{9}+y^2=1$, the x-axis, and the line $y=2x/3$ is equal to the area in the first quadrant enclosed by the ellipse $\frac{x^2}{9}+y^2=1$, the y-axis, and the line $y=mx$.

Problem. (1994 B-2) Find all c such that the graph of the function $x^4+9x^3+cx^2+ax+b$ meets some line in four distinct points.

Problem. (1995 A-2) For what pairs (a,b) of positive real numbers does the improper integral

$$\int_b^\infty \left(\sqrt{\sqrt{x+a}-\sqrt{x}}-\sqrt{\sqrt{x}-\sqrt{x-b}}\right)dx$$

converge?

Problem. (1995 B-2) An ellipse, whose semi-axes have lengths a and b, rolls without slipping on the curve $y=c\sin\left(\frac{x}{a}\right)$. How are a,b,c related, given that the ellipse completes one revolution when it traverses one period of the curve?

Problem. (1996 A-2) Let C_1 and C_2 be circles whose centers are 10 units apart, and whose radii are 1 and 3. Find, with proof, the locus of all points M for which there exists points X on C_1 and Y on C_2 such that M is the midpoint of the line segment XY.

Problem. (1996 B-2) Show that for every positive integer n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1\cdot 3\cdot 5\cdots(2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

Problem. (1997 A-2) Players $1,2,3,\ldots,n$ are seated around a table, and each has a single penny. Player 1 passes a penny to player 2, who then passes two pennies to player 3. Player 3 then passes one penny to Player 4, who passes two pennies to Player 5, and so on, players alternately passing one penny or two to the next player who still has some pennies. A player who runs out of pennies drops out of the game and leaves the table. Find an infinite set of numbers n for which some player ends up with all n pennies.

Problem. (1997 B-2) Let f be a twice-differentiable real-valued function satisfying

$$f(x)+f''(x)=-xg(x)f'(x),$$

where $g(x)\geq 0$ for all real x. Prove that $|f(x)|$ is bounded.

Problem. (1998 A-2) Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A+B$ depends only on the arc length, and not on the position, of s.

Problem. (1998 B-2) Given a point (a,b) with $0<b<a$, determine the minimum perimeter of a triangle with one vertex at (a,b), one on the x-axis, and one on the line $y=x$. You may assume that a triangle of minimum perimeter exists.

Problem. (1998 A-2) Show that for some fixed positive n we can always express a polynomial with real coefficients which is nowhere negative as a sum of the squares of n polynomials.

Problem. (1999 B-2) $p(x)$ is a polynomial of degree n. $q(x)$ is a polynomial of degree 2. $p(x)=p''(x)q(x)$ and the roots of $p(x)$ are not all equal. Show that the roots of $p(x)$ are all distinct.

3. The 00s

Problem. (2000 A-2) Prove that there exist infinitely many integers n such that n, $n+1$, and $n+2$ are each the sum of two squares of integers.

Problem. (2000 B-2) Prove that the expression

$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$. [Here $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ and $\gcd(m,n)$ is the greatest common divisor of m and n.]

Problem. (2001 A-2) You have coins $C_1, C_2, ..., C_n$. For each k, coin C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

Problem. (2001 B-2) Find all pairs of real numbers (x, y) satisfying the system of equations

$$1/x + 1/(2y) = (x^2 + 3y^2)(3x^2 + y^2)$$
$$1/x - 1/(2y) = 2(y^4 - x^4)$$

Problem. (2002 A-2) Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

Problem. (2002 B-2) Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices. Two players play the following game: Each player, in turn, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex. Show that the player who signs first will always win by playing as well as possible.

Problem. (2003 A-2) Let $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ be nonnegative real numbers. Show that

$$(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \leq ((a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n))^{1/n}$$

Problem. (2003 B-2) Let n be a positive integer. Starting with the sequence $1, 1/2, 1/3, \ldots, 1/n$ form a new sequence of $n-1$ entries $3/4, 5/12, \ldots, (2n-1)/2n(n-1)$ by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbors on the second sequence to obtain a third sequence of $n-2$ entries and continue the final sequence produced consists of a single number x_n. Show that $x_n < (2/n)$.

Problem. (2004 A-2) For $i = 1, 2$ let T_i be a triangle with side lengths a_i, b_i, c_i, and area A_i. Suppose that $a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2$, and that T_2 is an acute triangle. Does it follow that $A_1 \leq A_2$?

Problem. (2004 B-2) Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}$$

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