MATH 488A: TEST 2

1. Test 2

Problem 1. Numbers $1, 2, 3, ..., 2005$ are written on a blackboard. Every now and then somebody picks two numbers a, b and replaces them by $a-1$, $b+1$. Is it possible that at some point all numbers on the blackboard are even?

Solution of Problem 1. No, it is not possible. To see this, note first that the sum of all the numbers on the board is invariant under changes of the type described, since $(a-1)+(b+1) = a+b$. Now observe that

$$\sum_{n=1}^{2005} n = \frac{2005 \cdot (2005+1)}{2} = 2005 \cdot 1003,$$

which is odd. Thus, the sum of all the numbers on the blackboard will always be odd.

If all the numbers on the blackboard were even, then their sum would also be even. Thus, it is not possible to make changes of the type described that make all the numbers even. □

Problem 2. Given 9 integers all of whose prime divisors are ≤ 6, show that there exist 2 of these integers whose product is a perfect square. Is this always true for 8 integers satisfying the same condition? If so, prove it. If not, give a counterexample.

Solution of Problem 2. Our 9 integers may be written $2^{a_i}3^{b_i}5^{c_i}$, where $i \in \{1, 2, 3, \ldots, 9\}$. We observe that $2^{a_i}3^{b_i}5^{c_i} \cdot 2^{a_j}3^{b_j}5^{c_j} = 2^{a_i+a_j}3^{b_i+b_j}5^{c_i+c_j}$ is a square iff a_i+a_j, b_i+b_j, and c_i+c_j are all even. Let Y be the set of three-tuples with entries in $\{\text{even}, \text{odd}\}$. Let $X = \{2^{a_i}3^{b_i}5^{c_i} \mid 1 \leq i \leq 9\}$. Let $f\colon X \to Y$ be the function that assigns to each number in X the tuple $(a_i \mod 2, b_i \mod 2, c_i \mod 2)$. (Think of X as the pigeons and Y as the holes.) We have $|X| = 9 > |Y| = 8$, so there are two integers $2^{a_i}3^{b_i}5^{c_i}$ and $2^{a_j}3^{b_j}5^{c_j}$ in X whose product $2^{a_i+a_j}3^{b_i+b_j}5^{c_i+c_j}$ has all exponents congruent to $0 \mod 2$ and this product is therefore a square.

The same claim is *not* true for 8 integers. To see this, just pick 8 integers whose exponent vectors mod 2 (the values of f) are all different. For example, we may take the set of integers X to be $\{1, 2, 3, 5, 2\cdot 3, 2\cdot 5, 3\cdot 5, 2\cdot 3\cdot 5\}$. □

Problem 3. Two rivers run parallel 2 miles apart. Two cities A and B lie between the rivers; each city is equidistant from the rivers and the cities are 3 miles apart. A scientist wishes to travel from A to B, collecting a sample of water from each river during his journey. What is the length of the shortest path he can follow? Justify your answer.

Solution of Problem 3. The length of the shortest path is 5 miles.

Let us suppose A is to the west of B and that the rivers run east-west, as in the diagram on the next page. Let A' be an imaginary city two miles due south of A and let B' be an imaginary city two miles due north of B. We claim that, without loss of generality, we may assume that the shortest path satisfying the given conditions hits the southern river first (after leaving A). If not, then it hits the northern river first and we can reflect the path in the line from A to B, which doesn't change its length and produces a shortest path which hits the southern river first.

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The next stop for the path must be the northern river, since going directly to B does not achieve anything. Further, the three sections of the path (A to the southern river, river to river, and northern river to B), must all be line segments, since otherwise we could shorten the path by making these portions line segments.

Now, reflecting the first and last segments of the path as shown in the diagram, we get a path from A' to B'. The length of the path from A' to B' is of course the same as the length of our original path from A to B. Since the shortest path from A' to B' is a line segment, we may simply compute the length of this line segment, which, by the pythagorean theorem, is $\sqrt{3^2 + 4^2} = 5$ miles. □

Problem 4. There are $2n$ people attending a meeting. Each person knows at least n other participants. Show that it is possible to accommodate the participants in n rooms so that each room is occupied by two participants who know each other. Hint: consider the largest number k such that $2k$ of the participants can be accommodated as required and show that $k = n$.

Solution of Problem 4. We consider the largest number k such that $2k$ of the participants can be accomodated as required. Assume that $k < n$, so that there are at least 2 people a and b who have not yet been accomodated. We may assume that a and b do not know any of the other unaccomodated people, since k is maximal. Thus a and b know only people in the set of $2k$ people who have already been accomodated.

Now suppose c and d are sharing a room (i.e. c and d are in the set of $2k$ people who have already been accommodated). If a knows c and b knows d, then we can put a with c and b with d, contradicting the maximality of k. Thus, if a knows c, b cannot know d.

Since a knows at least n people, this means that there are n people in this set of $2k$ which b cannot know. This means b knows at most $2k - n$ people. But if $k < n$, then $2k - n < n$, which is a contradiction since by hypothesis b knows at least n people.

It follows that $k = n$ and all the participants can be accomodated. □

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