MATH 488A: SOLUTION TO PROBLEM 3.2.7

1. Sample Solution

Problem. (3.2.7) Suppose we have an infinite chessboard that contains a positive integer in each square. If the value in each square is the average of its four neighbors to the north, south, west, and east, prove that all the values in all the squares are equal.

Solution of Problem 3.2.7. Since any nonempty set of positive integers has a least element, there is a minimum value m (of all the squares) which is achieved in some square, which has neighboring values a, b, c, and d. We know that m is the average of these four values, i.e. that

$$\frac{a+b+c+d}{4} = m. \tag{1}$$

We can rewrite Equation 1 as

$$(a-m)+(b-m)+(c-m)+(d-m) = 0. \tag{2}$$

Since m is the minimum value, each of a, b, c, and d is greater than or equal to m. Thus, the parenthesized quantities in Equation 2 are nonnegative. Since they sum to zero, each of them is individually zero. That is, $a = b = c = d = m$.

To summarize, we have shown (1) that m appears somewhere, and (2) that if m appears in a square, it appears in all four neighboring squares.

Now we put coordinates on the squares of the infinite board so that m is in the cell $(0, 0)$ and other cells are labeled by coordinates $(i, j) \in \mathbb{Z} \times \mathbb{Z}$. We will prove by induction on $|i + j|$ that the minimum value m appears in every cell.

Specifically, let $P(n)$ be the proposition "All squares (i, j) with $|i + j| < n$ contain m". Thus $P(1)$ says that m appears in $(0, 0)$; this is true because of our choice of coordinates, so we have the basis step.

Now note that our argument at the beginning of the proof shows that (2') if m appears in the square (i, j) then m also appears in the squares $(i-1, j)$, $(i+1, j)$, $(i, j-1)$, and $(i, j+1)$. This lets us complete the induction step. Suppose that $P(n)$ is true, and that (i, j) is a cell with $|i+j| < (n+1)$. If $|i+j| < n$, then m appears in (i, j) by the induction hypothesis. So we can assume that $|i + j| = n$, i.e. that either $i + j = n$, or that $i + j = -n$.

If $i + j = n$, then $(i - 1) + j = n - 1 < n$, so m appears in the square $(i - 1, j)$ by the induction hypothesis, and m also appears in (i, j) by (2') above.

If $i + j = -n$, then $(i + 1) + j = -(n - 1)$, so m appears in the square $(i + 1, j)$ by the induction hypothesis, and m also appears in (i, j) by (2') above.

This completes the proof of the induction step, so we have shown $P(n)$ is true for all n; that is, every cell contains m. □

Mathematics Department, Binghamton Univeristy, P. O. Box 6000, Binghamton, New York, 13902-6000
E-mail address: dikran@math.binghamton.edu

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