Math 525 Class 2 August 25.

Course web page (with notes)

people.math.binghamton.edu/dikran/525

There is a summary (almost complete) of the entire course.

More course summaries:

people.math.binghamton.edu/dikran/455/

has sumarries of 447 and 448

(easier versions of 501 and 502)

More about course summaries:

My summary of Ch 8:

A Euclidean Ring is a P.I.D.

A P.I.D. is a U.F.D.

First quiz: Monday, August 28.

Covers Section 7.1 : so, assignment.

read 7.1.

Questions of the type: What does this theorem say? What is the definition of ______ ?

I will not go into excessive detail.

Example: the definition of ring in 7.1.

Do not bother memorizing this definition.

Legitimate question: • Is $M_{2\times 2}(\mathbb{R})$ a division ring? (Answer: no.)

- Give an example of a division ring which is not a field?

- According to the definition in 7.1 is 0 a zero-divisor?

- Notice that $\deg(fg) = \deg(f) + \deg(g)$

for polynomials f & g. $\in \mathbb{R}[x]$.

What about the zero polynomial?

How does the book resolve this issue?

There is a super-important example in 7.1 which you could spend a long time studying: the ring of algebraic integers in a number field. The book confines its attention to the quadratic case:

~~We rewrite~~

Algebraic integer: $\alpha \in \mathbb{C}$ is an algebraic integer if it is a root of a <u>monic</u> polynomial with integer coefficients.

Example: $\sqrt{2}$ is an algebraic integer because it satisfies $1 \cdot x^2 - 2 = 0$.

The algebraic integers form a ring.

If you have a field, which is formed from $\mathbb{Q}$ by adjoining roots of ~~all~~ polynomials with rational coefficients, ~~then~~ the algebraic integers in this field are a ring:

The book considers extensions $\mathbb{Q}(\sqrt{D})$ and tells you explicitly what the ring of ~~inte~~ algebraic integers in this field is. (Assume D is square free.)

~~If $D \equiv 2 \bmod 2$~~

If $D \equiv 1$ (4) the ring is $\mathbb{Z}\left[\frac{1+\sqrt{D}}{2}\right]$

If $D \equiv 2, 3$ (4) the ring is $\mathbb{Z}[\sqrt{D}]$.

In particular $\frac{1+\sqrt{5}}{2}$ is an algebraic integer. (satisfying $x^2 - x - 1 = 0$).

Some of these are U.F.D.s and some are not!

<u>Example:</u> in $\mathbb{Z}[\sqrt{-5}]$ note $-5 \equiv 3 \bmod$

$$(1+\sqrt{-5})(1-\sqrt{-5}) = 1^2 - (\sqrt{-5})^2 = 6 = 2 \cdot 3.$$

Show that $1+\sqrt{-5}$, $1-\sqrt{-5}$, 2, 3 cannot be further factored in $\mathbb{Z}[\sqrt{-5}]$.

Rings of algebraic integers are <u>very</u> <u>close</u> to being UFDs in the following sense:

We can form a group using the <u>ideals</u> in such a ring.

Now in the ordinary integers the group constructed from ideals is the same as the group of nonzero elements of $\mathbb{Q}$ using mult.

There is a subgroup corresponding to principal ideals, and in $\mathbb{Z}$, this subgroup is everything.

In the case of the ring of integers in a number field, the quotient by the subgroup generated by principal ideals is <u>finite</u>.

"Finiteness of the Ideal Class Group"