Math 525 Class 4 August 30.

Correcting a mistake from last time.

<u>Fact 1:</u> $\mathbb{H} \cong \mathbb{H}^{op}$

Recall that for any ring R we can define a new ring R^{op} such that $x \cdot_{op} y = yx$.

If $r, s \in \mathbb{H}$ $r \cdot_{op} s = sr$.

The isomorphism between $\mathbb{H}$ and $\mathbb{H}^{op}$ is "bar".

$\overline{x+y} = \bar{x} + \bar{y}$

$\overline{xy} = \bar{x} \cdot_{op} \bar{y} = \bar{y}\bar{x}$

$\Big\}$ Shows "bar" is a ring hom. $\mathbb{H} \to \mathbb{H}^{op}$.

<u>Fact 2:</u>
There is a mapping $\mathbb{H} \times \mathbb{H} \to M_{4\times 4}(\mathbb{R})$.
defined by regarding $\mathbb{H}$ as $\mathbb{R}^4$
and writing $(r, s)\, v = r v \bar{s}$.

This mapping satisfies

$$(q,h)\left[(r,s)\ v\right] = (q,h)\ rv\bar{s}$$

$$= qrv\bar{s}\bar{h}$$

$$= qr\ v\ \overline{hs}$$

$$= \cancel{q r} (qr, hs)\ v.$$

This will later be used to show

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong M_{4\times 4}(\mathbb{R})$$

after defining $\otimes_{\mathbb{R}}$

Q: In 7.2 the "group ring" RG is defined. We ~~kn~~ know the group

$\cancel{Q_8}$ $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$.

We can form the group ring $\mathbb{R}Q_8$.

What is the relationship between $\mathbb{H}$ and $\mathbb{R}Q_8$?

Warning about rings: in complete generality (not a special kind of ring, e.g. P.I.D.)

a ring is roughly as general as a set.

Problem 24 find a unit of infinite order in $\mathcal{O}_{\mathbb{Q}(\sqrt{D})}$ for $D = 3, 5, 6, 7$.

$$N(a + b\sqrt{D}) = a^2 - Db^2$$

$a \in \mathbb{Q}$ $b \in \mathbb{Q}$.

How to think about ring homomorphisms (7.3)

- A homomorphism of waffles is a map of sets that preserves the waffle structure on the set.
- Think by analogy with groups (or vector spaces, or...)

In the context of groups or vector spaces there is a concept of the kernel (or "null-space")

If $T: V \to W$ is a linear trans. then you learned in linear algebra.

$T(V)$ is a subspace of W.

$$\frac{V}{\ker T} \cong T(V).$$

$\ker T$ is a subspace of V.

There will be some adaptation necessary:

In group theory you learned that the kernel of a homomorphism is a special subgroup ["normal"]

The kernel of a ring homomorphism is a not just a subring, it's an "ideal".

More about $\mathbb{H}$ and $\mathbb{R}Q_8$.

$+1, -1$ are distinct elements of $\mathbb{R}Q_8$.

In $\mathbb{R}Q_8$ $\quad 5(+1) - 2(-1) \neq 7(+1)$

because $(+1)$ and (-1) are distinct elts of the group Q_8

In $\mathbb{H}$ $\quad 5(+1) - 2(-1) = 7$

Another way to build intuition: a new object is defined: an "ideal".

Ask: in the rings I know, e.g. $\mathbb{R}[x]$, or $\mathbb{Q}$, or $M_{3\times 3}(\mathbb{R})$, what are the ideals?

These questions might be either trivial or hard to answer in full generality.

Maybe the answer can be looked up?

Dual group: $G^* = \mathrm{Hom}(G, \mathbb{C}^\times)$

for a finite abelian group A, we have a theorem

$$A \cong A^*$$

The proof is (conceptually):

$C \cong C^*$ for a cyclic group C

$(G \times H)^* \cong G^* \times H^*$.

Apply Fund. Thm. of Finite Ab. Gp.

Try to learn proofs at this conceptual level, rather than studying line-by-line.