Math 525 Class 38 December 1

Quiz solutions next page

Q: Let R be a Noetherian ring and M a finitely-generated R-module.

Is M necessarily Noetherian?

Note that M must be Noetherian if R is a PID because of the classification of f.g. modules over a PID.

Remark on quiz #3 and #4:

$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/6\mathbb{Z}$, so in #3

$$A \cong \underbrace{(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z})}_{\text{reorder summands}} \cong \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$$

Math 525 — Quiz 18 – December 1

Name: Solutions

1. If we regard $\mathbb{Z}[\sqrt{3}]$ as a $\mathbb{Z}$-module, is this module Noetherian? Why or why not?

$\mathbb{Z}[\sqrt{3}]$ is a free $\mathbb{Z}$-module with basis $\{1, \sqrt{3}\}$. It is finitely generated and therefore Noetherian.

2. If we regard $\mathbb{Q}$ as a $\mathbb{Z}$-module, is this module Noetherian? Why or why not?

$\mathbb{Q}$ is not a finitely-generated $\mathbb{Z}$-module and therefore NOT Noetherian.

3. Define a $\mathbb{Z}$-module A by

$$A = (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z})$$

What is the invariant factor decomposition of A?

$\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$

4. Define a $\mathbb{Z}$-module A by

$$A = \mathbb{Z}/15\mathbb{Z} \oplus \mathbb{Z}/45\mathbb{Z}$$

What is the primary decomposition of A?

$$\underbrace{\left(\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}\right)}_{\text{3-primary}} \oplus \underbrace{\left(\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}\right)}_{\text{5-primary}}$$

Example: over $\mathbb{R}[x]$ consider the module

$$A = \mathbb{R}[x]/(x) \oplus \mathbb{R}[x]/(x) \oplus \mathbb{R}[x]/(x-1) \oplus \mathbb{R}[x]/(x-1)$$

$$\cong \mathbb{R}[x]/((x-1)\cdot x) \oplus \mathbb{R}[x]/((x-1)\cdot x) \quad (\star)$$

because $\mathbb{R}[x]/((x-1)\cdot x) \cong \mathbb{R}[x]/(x-1) \oplus \mathbb{R}[x]/(x)$

by CRT.

$(\star)$ is the invariant factor decomposition.

This is an analog of #3 on quiz.

In (4) we use $\mathbb{Z}/(3) \oplus \mathbb{Z}/(5) \cong \mathbb{Z}/(15)$

and $\mathbb{Z}/(9) \oplus \mathbb{Z}/(5) \cong \mathbb{Z}/(45)$

$$A \cong (\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}) \oplus (\mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z})$$

$$\cong (\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}) \oplus (\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z})$$

Why is $\mathbb{Z}/45 \cong \mathbb{Z}/5 \oplus \mathbb{Z}/9$?

Answer : CRT. (See corollary 18, p. 267)

(Note the requirement that the primes are distinct.)

$45 = 3^2 \cdot 5$, so $\mathbb{Z}/45 \cong \mathbb{Z}/3^2 \oplus \mathbb{Z}/5$

but we can't write $45 = 3 \cdot 3 \cdot 5$

and say $\mathbb{Z}/45 \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/5$

because 3 and 3 are NOT distinct.

Why be interested in Rational Canonical Form?

Suppose we want to understand the representation theory of $GL_3(\mathbb{F}_2)$. The first step is to find the conjugacy classes. Rational Canonical Form classifies matrices up to conjugacy. So if X is any matrix in $GL_3(\mathbb{F}_2)$ we know that the char. polyn. of X has degree 3.

And we know the possible orders of X

because $|GL_3(\mathbb{F}_2)| = 168 = (2^3-1)(2^3-2)(2^3-2^2)$

So if we have (say) $X^3 = I$ using the Cayley-Hamilton

theorem. And we have a short list of

canonical forms.