Math 525 Class 41 December 8

Companion Matrix: $a(x) = x^k + b_{k-1}x^{k-1} + \dots + b_1 x + b_0 \in F[x]$

$C_{a(x)}$ is the $k \times k$ matrix in $M_{k \times k}(F)$

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -b_0 \\ 1 & 0 & & \vdots & -b_1 \\ 0 & 1 & & & \vdots \\ \vdots & 0 & & & \vdots \\ & \vdots & & 0 & \\ 0 & 0 & & 1 & -b_{k-1} \end{pmatrix}$$

and $\det(xI - C_{a(x)}) = a(x)$.

Theorem 16: If $A \in M_{n \times n}(F)$

A is similar to a block diagonal matrix whose diagonal blocks are the companion matrices for monic polynomials $a_1(x) \cdots a_m(x)$ with.

$a_1(x) \mid a_2(x) \mid \cdots \mid a_m(x)$.

This form is unique and $a_m(x)$ is the minimal polynomial of A. The characteristic polynomial of A is the product of the a_i.

Application: lots! We looked at classification of elements in $M_{3\times 3}(\mathbb{Q})$ with $A^6 = I$, and at conjugacy classes of $GL_3(\mathbb{F}_2)$.

Recall the Fundamental Theorem of Linear Maps (a.k.a. "Rank - Nullity Theorem")

An $m \times n$ matrix A may be multiplied on the left and right by invertible matrices P, Q.
$P \in M_{m\times m}(F)$, $Q \in M_{n\times n}(F)$ $A \in M_{m\times n}(F)$
so that

$$PAQ = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \\ & & & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

rank(A) 1s.

Over a PID this generalizes to "Smith Normal Form". (Theorem 21) [which is not the most general possible]

$A \in M_{n\times n}(F)$. Using elementary row operations and column operations, $xI - A$ can be put in the diagonal form

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \\ & & & a_1(x) & & \\ & 0 & & & \ddots & \\ & & & & & a_m(x) \end{pmatrix}$$

$a_1(x) \mid a_2(x) \mid \cdots \mid a_m(x)$.

More generally, if R is a PID and $A \in M_{m\times n}(R)$ there are invertible matrices $P, Q \in M_{m\times m}(R)$ $M_{n\times n}(R)$ such that

$$PAQ = \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & 0 & \\ & & a_r & & & \\ & & & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{pmatrix} \qquad r = \text{rank}(A). \qquad a_1 | a_2 | \cdots | a_r$$

Applications: A homomorphism of free modules over a PID $R^n \xrightarrow{A} R^m$ may be, by changing basis, put in a form where it is just multiplication by an element of R on each basis element, analogous to FTLM in Axler.

Application 2: Hadamard matrices are $n\times n$ matrices of ± 1 with orthogonal rows (and hence orthogonal columns)

We will show 4 different 16×16 Hadamard matrices. Can these matrices be transformed into one another by permuting rows and/or columns?

Note that permuting rows and columns is a special case of "invertible linear transformations".

suitable row and column permutations to it), in practice this is not feasible if the matrices are even moderately large. A negative criterion is available from the observation that matrices that are permutation equivalent must have the same Smith normal form. This is quite a useful criterion, and with it we can show, for example, that the following four 16×16 Hadamard matrices H_1, H_2, H_3, H_4, which were produced by Marshall Hall, Jr., and which represent (up to permutation equivalence) any 16×16 Hadamard matrix, are in fact not permutation equivalent, since they are not even equivalent (in the Smith sense):

$$H_1 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1
\end{pmatrix},$$

$$H_2 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1
\end{pmatrix},$$

$$H_3=\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&-1&-1&-1&-1&-1&-1&-1&-1\\
1&1&1&1&-1&-1&-1&-1&1&1&1&1&-1&-1&-1&-1\\
1&1&1&1&-1&-1&-1&-1&-1&-1&-1&-1&1&1&1&1\\
1&1&-1&-1&1&1&-1&-1&1&1&-1&-1&1&1&-1&-1\\
1&1&-1&-1&1&1&-1&-1&-1&-1&1&1&-1&-1&1&1\\
1&1&-1&-1&-1&-1&1&1&1&1&-1&-1&-1&-1&1&1\\
1&1&-1&-1&-1&-1&1&1&-1&-1&1&1&1&1&-1&-1\\
1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1\\
1&-1&1&-1&1&-1&1&-1&-1&1&-1&1&-1&1&-1&1\\
1&-1&1&-1&-1&1&-1&1&1&-1&-1&1&1&-1&-1&1\\
1&-1&1&-1&-1&1&-1&1&-1&1&1&-1&-1&1&1&-1\\
1&-1&-1&1&1&-1&-1&1&1&-1&-1&1&-1&1&1&-1\\
1&-1&-1&1&1&-1&-1&1&-1&1&1&-1&1&-1&-1&1\\
1&-1&-1&1&-1&1&1&-1&1&-1&1&-1&-1&1&-1&1\\
1&-1&-1&1&-1&1&1&-1&-1&1&-1&1&1&-1&1&-1
\end{pmatrix},$$

$$H_4=\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&-1&-1&-1&-1&-1&-1&-1&-1\\
1&1&1&1&-1&-1&-1&-1&1&1&1&1&-1&-1&-1&-1\\
1&1&1&1&-1&-1&-1&-1&-1&-1&-1&-1&1&1&1&1\\
1&1&-1&-1&1&1&-1&-1&1&1&-1&-1&1&1&-1&-1\\
1&1&-1&-1&1&1&-1&-1&-1&-1&1&1&-1&-1&1&1\\
1&1&-1&-1&-1&-1&1&1&1&1&-1&-1&-1&-1&1&1\\
1&1&-1&-1&-1&-1&1&1&-1&-1&1&1&1&1&-1&-1\\
1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-1\\
1&-1&1&-1&1&-1&-1&1&1&-1&-1&1&-1&1&-1&1\\
1&-1&1&-1&-1&1&1&-1&-1&1&-1&1&1&-1&-1&1\\
1&-1&1&-1&-1&1&-1&1&-1&1&1&-1&-1&1&1&-1\\
1&-1&-1&1&1&-1&1&-1&-1&1&-1&1&-1&1&1&-1\\
1&-1&-1&1&1&-1&-1&1&-1&1&1&-1&1&-1&-1&1\\
1&-1&-1&1&-1&1&1&-1&1&-1&1&-1&-1&1&-1&1\\
1&-1&-1&1&-1&1&-1&1&1&-1&-1&1&1&-1&1&-1
\end{pmatrix}.$$

The invariant factors of these matrices are given by

$$H_1\colon\ 1, 2, 2, 2, 2, 4, 4, 4, 4, 4, 4, 8, 8, 8, 8, 16,$$
$$H_2\colon\ 1, 2, 2, 2, 2, 2, 4, 4, 4, 4, 8, 8, 8, 8, 8, 16,$$
$$H_3\colon\ 1, 2, 2, 2, 2, 2, 2, 4, 4, 8, 8, 8, 8, 8, 8, 16,$$
$$H_4\colon\ 1, 2, 2, 2, 2, 2, 2, 2, 8, 8, 8, 8, 8, 8, 8, 16.$$

Looking at these factors, we see, for example, that the number of 2's in each of the four lists is different. Hence our matrices are inequivalent, and therefore also permutation inequivalent.

so if such a permutation were possible, the matrices would have the same smith normal form.

So, compute Smith normal forms, see that they are different, and conclude the matrices cannot be transformed into one another by permutation of rows and/or columns.

Jordan Form:

A Jordan block (of size k with eigenvalue λ.)

$$\begin{pmatrix} \lambda & 1 & 0 & & 0 \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & 0 \\ & & & \ddots & 1 \\ 0 & & & & \lambda \end{pmatrix}$$

a $k \times k$ matrix with λ on the main diagonal 1 along the first superdiagonal, 0 elsewhere.

This block has minimal polyn. $(x-\lambda)^k$ and characteristic polyn $(x-\lambda)^k$.

If F contains all eigenvalues of $A \in M_{n \times n}(F)$, then A can be put into block diagonal form where every block on the diagonal is a Jordan block.

This is unique up to permutation of blocks.

Two things to understand about Jordan blocks.
(Simplest case!)

$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ is not diagonalizable!

The eigenvalues are the roots of the characteristic polynomial, so for an upper Δ matrix they are the diagonal elements.

Thus if $PAP^{-1} = D$, diagonal, $D = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

Note $D - \lambda I = 0$, so we would have

$$P(A - \lambda I)P^{-1} = PAP^{-1} - \lambda I = 0$$

But $A - \lambda I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Jordan blocks NOT diagonalizable.

We can change basis so that $\underbrace{\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}}_{T}$ becomes

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in the new basis.

Let $T(e_1) = e_1$ $T(e_2) = e_2 + 2e_1$.

Now let $f_1 = 2e_1$ $f_2 = e_2$. We have

$T(f_1) = T(2e_1) = 2e_1 = f_1$ $T(f_2) = f_2 + 2e_1 = f_2 + f_1$.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad A^3 = O.$$

$Ae_1 = O \quad Ae_2 = e_1 \quad Ae_3 = e_2$

e_i = standard basis vectors.

$e_3 \xrightarrow{A} e_2 \xrightarrow{A} e_1 \xrightarrow{A} O$ (Schematic - not a real notation)

If J is the Jordan $k \times k$ block of the previous page $(J - \lambda I)^k = O$.

$(J - \lambda I)^{k-1} \neq O$.

Why do we have ones?

Why is $\begin{pmatrix} \lambda & 2 & 0 \\ 0 & \lambda & 2 \\ 0 & 0 & \lambda \end{pmatrix}$ not a possible Jordan Form?

Using this idea we can change basis

So that $\begin{pmatrix} \lambda & \alpha & 0 \\ 0 & \lambda & \beta \\ 0 & 0 & \lambda \end{pmatrix}$ becomes $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$

and more generally any Jordan block can be ~~the~~ transformed by change of basis so that it's just ones on the "superdiagonal."

What good is Jordan Form?

Shows that certain exponential maps from Lie Algebras to Lie Groups are surjective.

Ex 2, p. 499: If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A then $\lambda_1^k, \ldots, \lambda_n^k$ are the eigenvalues of A^k.

Note: this requires $\lambda_1, \ldots, \lambda_n \in F$ (or field alg. closed because of std. example)

$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ rot $90°$ ⟲ in $\mathbb{R}^2$.

char polyn. of of A is x^2+1.

eigenvalues of A are $\pm i$

or: over $\mathbb{R}$ no eigenvalues.

$A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ eigenvalues -1.

of course $(\pm i)^2 = -1$.

But working over $\mathbb{R}$, eigenvalues of $A = \emptyset$.

and eigenvalues of $A^2 = \{-1\}$.

More applications of the Jordan Form:

You can prove the Cayley - Hamilton Theorem

A matrix satisfies its own characteristic polynomial.

(This isn't the ideal proof of Cayley-Hamilton)

If a matrix is in block diagonal form, the characteristic polyn. is the product of the characteristic polynomials for the blocks.

We see from the def'n of Jordan block that each Jordan block satisfies its own char. polyn.

So each Jordan block satisfies the overall characteristic polyn.

So when we apply ~~the char p~~ any polyn. to a matrix in block form.

$$f\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} f(A) & 0 \\ 0 & f(B) \end{pmatrix}$$

So any matrix in Jordan form satisfies its char. polyn.

Any matrix can be put in Jordan form

By conjugating by the change of basis matrix.

Conjugation does not change the char. polyn. or the fact that it evaluates to O.

$$\chi_A(x) = \det(xI - A).$$

$$\chi_{PAP^{-1}}(x) = \cancel{\det(PA(xI-}$$

$$= \det(xI - PAP^{-1})$$

$$= \det(P(xI - A)P^{-1})$$

$$= \det(P)\det(xI - A)\det(P^{-1}).$$

$$= \det(xI - A) = \chi_A(x).$$

If $f(A) = O \qquad f \in F[x]$.

$$Pf(A)P^{-1} = f(PAP^{-1}). = O$$

because e.g. $P(I + A^2)P^{-1} = I + PA^2P^{-1}$

$$= I + (PAP^{-1})(PAP^{-1}).$$