Math 525 Class 26 Nov. 1

Mistake in test:

$\mathbb{Z}[x^2]$ is the ring of polynomials in x^2

or: polynomials in which only even powers of x appear.

Test 1, Problem 9: question only asks "How many".

You may quote Problem 5(a) p. 267 without pf.

You do NOT need to find the n.

Theorem 8. p. 246 part (3) (I)

Example: $6\mathbb{Z}$ is an ideal of $\mathbb{Z}$. $\supset A$.

Corresp. $A \longleftrightarrow A/I$ between subrings A of $\mathbb{Z}$ containing $I = 6\mathbb{Z}$ and subrings of $\mathbb{Z}/6\mathbb{Z}$.

Subrings of $\mathbb{Z}/6\mathbb{Z}$ $= \{0,1,2,3,4,5\}$

$\mathbb{Z}/6\mathbb{Z}$, 0, $\{0,2,4\}$, $\{0,3\}$

Subrings of $\mathbb{Z}$ containing $6\mathbb{Z}$

$\mathbb{Z}$, $6\mathbb{Z}$, $2\mathbb{Z}$, $3\mathbb{Z}$.

Example: $\mathbb{F}_2[x]$ $\quad I = (x^2+x+1)$.

$$\mathbb{F}_2[x]/(x^2+x+1) = \{ f + I \mid f \in \mathbb{F}_2[x] \}.$$

~~Check see~~ The zero element of the quotient is the coset $0 + I$. So any multiple of x^2+x+1, say $h\cdot(x^2+x+1) + I = \underset{+I}{0} \in \mathbb{F}_2[x]/(x^2+x+1)$

So we need only consider cosets of the form $(a+bx) + I$ because we can get rid of higher terms by ~~ad~~ adding multiple of (x^2+x+1).

If $R = \mathbb{F}_2[x]/(x^2+x+1)^2 \leftarrow J$

We can consider cosets of the form

$a+bx+cx^2+dx^3 + J$ because

by adding a multiple of $(x^2+x+1)^2$ we can get rid of all 4^{th} power terms.

Idea: $a+bx+cx^2+dx^3$ can be written

$(a+bx) + (e+fx)(x^2+x+1)$.

When is this elt. invertible in R?

Quiz on 10.4

You should know: the universal property of the tensor product.

What type of object is the tensor product?

- If R is a comm ring then R-modules M, N are two-sided and $M \otimes_R N$ is an R-module.

- If R is NOT comm. then from a right R-mod M and left R-mod N we can form $M \otimes_R N$

this is an ab. gp.

Why is this?

We want the tensor product $M \otimes_R N$

to be generated by symbols $m \otimes n$

with bilinearity properties. (over R).

$$m \otimes n + m' \otimes n = (m+m') \otimes n$$

$$m \otimes n + m \otimes n' = m \otimes (n+n')$$

$$ma \otimes n = m \otimes an.$$

We can't multiply $a \cdot (m \otimes n)$

because $am \otimes n$ has no meaning.

M is only a right R-module.

The formal def'n of $\otimes_R$ is

take the free ~~module on~~ $\mathbb{Z}$-module on $M \times N$.

then take quotient by the submodule gen. by. all elts of the following forms:

$(m,n) + (m',n) - (m+m',n)$

$(ma,n) - (m,an)$ etc.

for all $m, m' \in M$ $n \in N$ $a \in R$.

Don't work with this def'n if you can avoid it!

If we do this with a comm ring R. then we have to alter the def'n. to get $M \otimes_R N$ to be an R-module.

We start with the free R-mod on $M \times N$.

$$\begin{tikzcd} M \times N \arrow[r, "\text{incl}"] \arrow[dr, "\varphi"'] & M \otimes_R N \arrow[d, dotted, "\exists!\ \Phi"] \\ & X \end{tikzcd}$$

φ: R-bilinear map to X.

$\text{incl}(m, n) = m \otimes n$

Motivation for tensor product.

We have a FDVS over $\mathbb{R}$ with some basis $e_1, \dots, e_n$. "$V \cong \mathbb{R}^n$"

Create a new FDVS / $\mathbb{C}$ by saying the basis is $e_1, \dots, e_n$ and allowing $\mathbb{C}$-multiples of these basis vectors. "$\mathbb{C} \otimes_{\mathbb{R}} V$"

How can we do this in a basis-free way?

$\mathbb{C} \otimes_{\mathbb{R}} V \cong \mathbb{C}^n$.

Strange Properties of $\otimes$ follow from

bilinearity: $\left(\mathbb{Z}/2\mathbb{Z}\right) \otimes_{\mathbb{Z}} \left(\mathbb{Z}/3\mathbb{Z}\right) \cong \{0\}$.

bilinearity shows: $1\cdot 2 \otimes 1 = 0 \otimes 1 = 0$-object in $\otimes$.

$$\downarrow$$

$$1 \otimes 2\cdot 1 = 1 \otimes 1$$

Even worse: $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \cong \{0\}$.

For Quiz: Focus on universal prop, what type of object, how do you define $\otimes$.