Math 525 Class 27 Nov 3.

Quiz solutions next page.

Also a bonus quiz!

#8 on the bonus quiz is a special case of Horn - $\otimes$ interchange, which itself is a special case of adjoint functors.

What is "adjoint"?

In linear algebra, if V is an IPS and $A: V \to V$ is a linear transf then the linear transformation A^* defined by

$$\langle v, Aw \rangle = \langle A^* v, w \rangle$$

is called the "adjoint" of A.

Think now about free groups. If X is a set. we can form a group $F(X)$ - free group on X.

If G is a group we can form a set $U(G)$

Math 525 — Quiz 12 – November 3

Name: Solutions

Let R be a ring (not necessarily commutative), M a right R-module, and N a left R-module.

1. What type of object is the tensor product $M \otimes_R N$? an abelian group

2. What is the *definition* of the tensor product $M \otimes_R N$?

We take the free $\mathbb{Z}$-module on $M \times N$ and then take the quotient by the submodule generated by all elements of the following 3 forms.

$(m,n) + (m',n) - (m+m',n)$ $\quad \forall m,m' \in M \ \forall n \in N$

$(m,n) + (m,n') - (m,n+n')$ $\quad \forall m \in M \ \forall n,n' \in N$

$(mr,n) - (m,rn)$ $\quad \forall m \in M \ \forall n \in N \ \forall r \in R.$

$M \otimes_R N$ has a universal property with respect to maps $\phi\colon M \cancel{\otimes_R} \times N \to L$.

3. In the statement of the universal property, what kind of object is L? an abelian gp.

4. In the statement of the universal property, what kind of map is ϕ?

an "R-balanced" map

5. What is the universal property of the tensor product $M \otimes_R N$?

$$\begin{tikzcd} M \times N \arrow[r, "\text{"incl"}"] \arrow[dr, bend right, "\phi"'] & M \otimes_R N \arrow[d, dotted, "\exists!\, \Phi"] \\ & L \end{tikzcd}$$

For every ab gp L and R-balanced ϕ, $\exists!\, \Phi$ hom of ab gps, making the diagram commute.

Math 525 — Quiz 22 – November 2

Name:______________________________

The following tensor products are isomorphic to familiar and well-known abelian groups. Using the results about tensor products in Section 10.4, describe the given tensor products in simpler terms, i.e. not as tensor products.

1. $\mathbb{R} \otimes_{\mathbb{Z}} (\mathbb{Z}^2)$ $\cong \mathbb{R}^2$

2. $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/8\mathbb{Z})$ $\cong 0$

3. $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^3$ $\cong \mathbb{R}^6$

4. $(\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/5\mathbb{Z})$ $\cong 0$

5. $(\mathbb{Z}/15\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/39\mathbb{Z})$ $\cong \mathbb{Z}/3\mathbb{Z}$

6. $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})$ $\cong 0$

7. $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ $\cong \mathbb{Q}$

8. $V^* \otimes_{\mathbb{R}} W$, where V, W are finite-dimensional real vector spaces and V^* is the dual space of V.

$$\cong \operatorname{Hom}_{\mathbb{R}}(V, W) = \{ \text{linear trans. } V \to W \}$$

An element of $V^* \otimes W$ is

$f_1 \otimes w_1 + \cdots + f_n \otimes w_n$ for $f_i \in V^*$, $w_i \in W$.

Define a map $V \to W$ by

$\phi(v) = f_1(v) w_1 + \cdots + f_n(v) w_n$ and verify

that this is a linear trans.

the underlying set.

If X, Y are two sets, $\mathrm{Hom}_{Set}(X, Y)$ is all functions $X \to Y$.

If G, H are groups, $\mathrm{Hom}_{Group}(G, H)$ is all group homomorphisms $G \to H$.

Now suppose we study $\mathrm{Hom}_{Set}(X, u(G))$

this is the same as $\mathrm{Hom}_{Group}(F(X), G)$

$$\mathrm{Hom}_{Set}(X, u(G)) = \mathrm{Hom}_{Group}(F(X), G)$$

says u and F are adjoint functors.

What is the relevance to #8 on the bonus quiz?

$$V^* = \mathrm{Hom}(V, \mathbb{R})$$

#8 says.
$$V^* \otimes W = \mathrm{Hom}(V, \mathbb{R}) \otimes W$$
$$= \mathrm{Hom}(V, W)$$
$$= \mathrm{Hom}(V, \mathbb{R} \otimes W).$$

There is an adjoint functor thing going on here too.

Theorem 43 "Adjoint Associativity".

R, S rings $\quad A$ right R-mod.

B (R, S) bimodule.

C right S-module

$$\mathrm{Hom}_S(A \otimes_R B, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$$

$- \otimes_R B$ is adjoint to $\mathrm{Hom}_S(B, -)$

U: Groups $\longrightarrow$ Sets $\qquad F$: Sets $\longrightarrow$ Groups

We will have another quiz on 10.4, and goes to "bonus quiz" on Monday Nov 6^{th}