

Quiz solutions next page.

Also a bonus quiz!

#8 on the bonus quiz is a special case of Horn - $\otimes$ interchange, which itself is a special case of adjoint functors.

What is "adjoint"?

In linear algebra if V is an IPS and $A: V \rightarrow V$ is a linear transfr then the linear transformation A^* defined by.

$$\langle v, Aw \rangle = \langle A^*v, w \rangle$$

is called the "adjoint" of A .

Think now about free groups. If X is a set.

we can form a group $F(X)$ - free group on X .

If G is a group we can form a set $u(G)$

Name: Solutions

Let R be a ring (not necessarily commutative), M a right R -module, and N a left R -module.

1. What type of object is the tensor product $M \otimes_R N$? *an abelian group*

2. What is the definition of the tensor product $M \otimes_R N$?

We take the free $\mathbb{Z}$ -module on $M \times N$ and then take the quotient by the submodule generated by all elements of the following 3 forms.

$$\begin{aligned} (m, n) + (m', n) - (m+m', n) & \quad \forall m, m' \in M \quad \forall n \in N \\ (m, n) + (m, n') - (m, n+n') & \quad \forall m \in M \quad \forall n, n' \in N \\ (mr, n) - (m, rn) & \quad \forall m \in M \quad \forall n \in N \quad \forall r \in R. \end{aligned}$$

$M \otimes_R N$ has a universal property with respect to maps $\phi: M \otimes_R N \rightarrow L$.

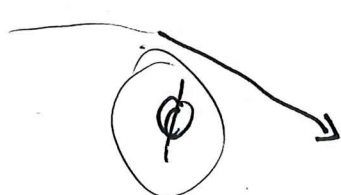
3. In the statement of the universal property, what kind of object is L ? *an abelian gp.*

4. In the statement of the universal property, what kind of map is ϕ ?

an "R-balanced" map

5. What is the universal property of the tensor product $M \otimes_R N$?

$$M \times N \xrightarrow{\text{"incl"}} M \otimes_R N$$



$\exists! \phi$

For every ab gr L and R -balanced ϕ , $\exists! \phi$ making the diagram commute.

Name: _____

The following tensor products are isomorphic to familiar and well-known abelian groups. Using the results about tensor products in Section 10.4, describe the given tensor products in simpler terms, i.e. not as tensor products.

1. $\mathbb{R} \otimes_{\mathbb{Z}} (\mathbb{Z}^2) \cong \mathbb{R}^2$

2. $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/8\mathbb{Z}) \cong 0$

3. $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^3 \cong \mathbb{R}^6$

4. $(\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/5\mathbb{Z}) \cong 0$

5. $(\mathbb{Z}/15\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/39\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$

6. $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}) \cong 0$

7. $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$

8. $V^* \otimes_{\mathbb{R}} W$, where V, W are finite-dimensional real vector spaces and V^* is the dual space of V .

$$\cong \text{Hom}_{\mathbb{R}}(V, W) = \{ \text{linear trans. } V \rightarrow W \}$$

An element of $V^* \otimes W$ is

$$f_1 \otimes w_1 + \dots + f_n \otimes w_n \quad \text{for } f_i \in V^*, w_i \in W.$$

Define a map $V \rightarrow W$ by

$$\phi(v) = f_1(v)w_1 + \dots + f_n(v)w_n \quad \text{and verify}$$

that this is a linear trans.

the underlying set.

If X, Y are two sets, $\text{Hom}_{\text{Set}}(X, Y)$ is all functions $X \rightarrow Y$.

If G, H are groups, $\text{Hom}_{\text{Group}}(G, H)$ is all group homomorphisms $G \rightarrow H$.

Now suppose we study $\text{Hom}_{\text{Set}}(X, \iota(G))$

this is the same as $\text{Hom}_{\text{Group}}(F(X), G)$

$$\text{Hom}_{\text{Set}}(X, \iota(G)) = \text{Hom}_{\text{Group}}(F(X), G)$$

says ι and F are adjoint functors.

What is the relevance to #8 on the bonus quiz? $V^* = \text{Hom}(V, R)$

#8 says. $V^* \otimes W = \text{Hom}(V, R) \otimes W$

$$= \text{Hom}(V, W)$$

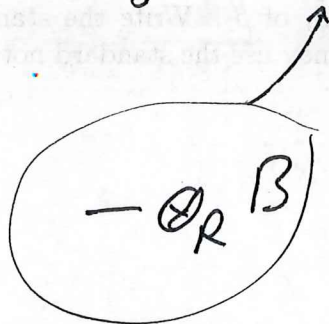
$$= \text{Hom}(V, R \otimes W).$$

There is an adjoint functor thing going on here too.

Theorem 43 "Adjoint Associativity"

R, S rings A right R -mod.
 B (R, S) bimodule.
 C right S -module

$$\text{Hom}_S(A \otimes_R B, C) \cong \text{Hom}_R(A, \text{Hom}_S(B, C))$$



is adjoint to $\text{Hom}_S(B, -)$

$U: \text{Groups} \rightarrow \text{Sets}$ $F: \text{Sets} \rightarrow \text{Groups}$

We will have another quiz on 10.4; analogous
to "bonus quiz" on Monday Nov 6th.