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Principle of group theory: if we define a subgroup without referring to "specific elements" of the group, that subgroup is normal.

Example: Let $Z = \{ z \in G \mid zg = gz \text{ for all } g \in G \}$

the "center" Z is a normal subgroup.

Non-example: let $D_8 = \langle a, b \mid a^4 = e, b^2 = e, bab = a^3 \rangle$

Let $H = \langle b \rangle$. H is <u>not</u> a normal subgroup of D_8. "b" is a specific element of D_8.

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Name: Solutions

The following tensor products are isomorphic to familiar and well-known abelian groups. Using the results about tensor products in Section 10.4, describe the given tensor products in simpler terms, i.e. not as tensor products.

1. $\mathbb{C} \otimes_{\mathbb{R}} (\mathbb{R}^2)$ $\mathbb{C}^2$

2. $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/12\mathbb{Z})$ 0

3. $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{Q}$ $\mathbb{Q}[i]$

4. $(\mathbb{Z}/10\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/6\mathbb{Z})$ $\mathbb{Z}/2\mathbb{Z}$

5. $\mathbb{Q}(i) \otimes_{\mathbb{Z}[i]} \mathbb{Q}(i)$ $\mathbb{Q}(i)$

6. $(\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})$ 0

7. $\mathbb{Z}^5 \otimes_{\mathbb{Z}} \mathbb{Z}^3$ $\mathbb{Z}^{15}$

8. The statement "Hom and $\otimes$ are adjoint functors" means that there is a (natural) isomorphism

$$\mathrm{Hom}(U \otimes V, W) \to \mathrm{Hom}(U, \mathrm{Hom}(V, W))$$

How do you define this isomorphism without referring to bases or specific elements of U, V, W? You may, if you like, assume that U, V, W are finite-dimensional vector spaces over the real numbers.

Given $\phi \in \mathrm{Hom}(U \otimes V, W)$ we can define $\theta \in \mathrm{Hom}(U, \mathrm{Hom}(V, W))$ by

$$\theta(u)(v) = \phi(u \otimes v)$$

Quiz #8 uses "specific elements" in a way analagous to its use in this principle of group theory.

If we say, "let $(e_1, \ldots, e_n)$ be a basis of V and define $f(a_1 e_1 + \cdots + a_n e_n) = a_1$ we have made reference to specific elements of V.

If we say instead something like "for all $v \in V$, define ... " we have NOT made reference to specific elements.

Quiz #8 solution is analgous to this second case.

What is a projective module?

Formally, a projective module is one which satisfies any of the 4 equivalent conditions of Prop 30.

Examples (and non-examples) of projective modules.

- Free modules are projective.
- $\mathbb{Z}/n\mathbb{Z}$ is NOT a projective $\mathbb{Z}$-module.
- $\mathbb{Q}$ is NOT a projective $\mathbb{Z}$-module.
- A finitely generated projective $\mathbb{Z}$-module must be free.

These examples don't show ~~no~~ nontrivial (i.e. non-free) projective modules.

Note condition (4) is that

P is projective iff it is a direct summand of a free module.

Here are some big classes of projective modules:

- Let R be a Dedekind ring, e.g. the ring of algebraic integers in a number field. Let $I \subset R$ be an ideal. Then I is a projective R-module.

- Vector bundles are projective modules.

- Let G be a finite group and F a field of characteristic 0, e.g. $\mathbb{Q}, \mathbb{R}, \mathbb{C}$. Then any FG-module M is projective.

These examples can be non-free.

How do we know that these are projective?

More about this in later chapters.

One specific case: $R = \mathbb{Z}[\sqrt{-5}]$

$I = (2, \alpha) \qquad \alpha = 1 + \sqrt{-5}.$

$J = (\alpha, 3)$

Claim $I \cong J$ as R-modules.

$I \xrightarrow{\cdot \frac{\alpha}{2}} J$ is an isomorphism

because $2 \cdot \frac{\alpha}{2} = \alpha \quad \frac{\alpha \cdot \alpha}{2} = \frac{6}{2} = 3.$

To prove that I is a direct summand of a free module, it suffices to prove that $I \oplus J \cong R \oplus R$.

~~Xa~~ To do this, we write down an exact sequence

$$0 \to I \cap J \xrightarrow{\phi} I \oplus J \xrightarrow{+} R \to 0$$

Note $+$ is onto $-2 + 3 = 1 \in R$.

$$I \cap J \longrightarrow I \oplus J$$

$$\phi(x) = (-x, x) \in I \oplus J.$$

$I \cap J = (\alpha)$. (Needs a little calculation)

Thus, $I \cap J \cong R$.

Since R is free, this exact sequence shows that $I \oplus J \cong R \oplus R$

and therefore I is a projective R-module.

This can be vastly generalized, see section 16.3.

The concept of "projective module" is used frequently in topology and there are many topological analogies that may help.

In particular, the "lifting property"