Math 525 Class 29 November 8.

Homework due date postponed to Monday Nov $13^{th}$.

Quiz Friday Nov $10^{th}$ on 10.5.

There are 3 types of modules discussed:

Projective, Injective, and Flat.

For the quiz: know concretely what these modules are in the case where the ring is $\mathbb{Z}$ (or maybe a PID)

There is also the terminology of "exact ~~seq~~ sequences."
This definition enormously simplifies a wide range of arguments. If you use this definition in the obvious way, you will have one simple thing to do at every stage of your argument and the result will almost prove itself.

Example: 5-Lemma.

An important example to know:

The exact sequence: $0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \to 0$

What happens to this exact sequence under various transformations: For example, suppose $A$ is a $\mathbb{Z}$-module.

$M \xrightarrow{f} N$ is a $\mathbb{Z}$-module map.

$$\mathrm{Hom}(A, M) \longrightarrow \mathrm{Hom}(A, N)$$

by composition $\quad A \xrightarrow{\phi} M \xrightarrow{f} N.$

$$\cancel{\phi(f)} \; f(\phi(\cdot)) : A \to N$$

$\mathrm{Hom}(A, -)$ is a transformation which we could apply to an exact sequence.

Also $- \otimes A$ is a transformation we could apply.

Also $\mathrm{Hom}(-, A)$ is an arrow-reversing transformation.

What happens to $0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$

When we apply $\mathrm{Hom}(-, \mathbb{Z}/p\mathbb{Z})$ ?

if $M \xrightarrow{f} N$ is a $\mathbb{Z}$-module map.

then $\mathrm{Hom}(M, \mathbb{Z}/p\mathbb{Z}) \longleftarrow \mathrm{Hom}(N, \mathbb{Z}/p\mathbb{Z})$.

$$M \xrightarrow{f} N \xrightarrow{\phi} \mathbb{Z}/p\mathbb{Z}$$

$$\phi(f(\cdot)) : M \to \mathbb{Z}/p\mathbb{Z} \in \mathrm{Hom}(M, \mathbb{Z}/p\mathbb{Z}).$$

The groups in the exact seq become.

$$\begin{tikzcd}
 & \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \arrow[d, phantom, "\cong"] & \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \arrow[l] \arrow[d, phantom, "\cong"] & \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \arrow[l] \arrow[d, phantom, "\cong"] & \\
0 & \mathbb{Z}/p\mathbb{Z} \arrow[l] & \mathbb{Z}/p\mathbb{Z} \arrow[l, "g"'] & \mathbb{Z}/p\mathbb{Z} \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

here

Suppose $\phi \in \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$. What is $g(\phi)$?

$g(\phi)$ is $\mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \xrightarrow{\phi} \mathbb{Z}/p\mathbb{Z}$

Thus $g(\phi) = \cancel{0 \in \mathrm{Hom}}\ \phi(p \cdot \_) = 0$.

What about the other map?

$$\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \xleftarrow{h} \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$$

Defined by composition. Let $\theta \in \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$

$h(\theta)$ is $\quad \mathbb{Z} \xrightarrow{\text{red}} \mathbb{Z}/p\mathbb{Z} \xrightarrow{\theta} \mathbb{Z}/p\mathbb{Z}$.

We know that the source and target of $h$ are $\cong \mathbb{Z}/p\mathbb{Z}$. What is $h$?

A: $h$ is $id_{\mathbb{Z}/p\mathbb{Z}}$.

So our exact sequence became:

$$0 \leftarrow \mathbb{Z}/p\mathbb{Z} \xleftarrow{0} \mathbb{Z}/p\mathbb{Z} \xleftarrow{id} \mathbb{Z}/p\mathbb{Z} \leftarrow 0.$$

This is no longer an exact sequence.

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Flat modules: these have the property that tensoring preserves exactness.

With our standard example:

$$0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \xrightarrow{\text{red}} \mathbb{Z}/p\mathbb{Z} \to 0.$$

We can take $- \otimes \mathbb{Z}/p\mathbb{Z}$ and get.

$$0 \to \mathbb{Z}/p\mathbb{Z} \xrightarrow{0} \mathbb{Z}/p\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}/p\mathbb{Z} \to 0.$$

not exact here, hence $\mathbb{Z}/p\mathbb{Z}$ is NOT a flat $\mathbb{Z}$-module.

Taking $- \otimes \mathbb{Z}$ we get the same sequence back. This holds generally: $A \otimes_{\mathbb{Z}} \mathbb{Z} \cong A$.

and $\mathbb{Z}$ is a flat $\mathbb{Z}$-module

More generally, free modules are flat.

Also: projective modules are flat.

We noted that over $\mathbb{Z}$, "projective" and "free" are the same thing.

So to get interesting examples of ~~vector bundles~~ projective modules

We needed to move to more general rings.

(Example was $\mathbb{Z}[\sqrt{-5}]$)

A non-free flat module over $\mathbb{Z}$ is $\mathbb{Q}$.

Notice that $\mathbb{Q}$ is NOT a f.g. $\mathbb{Z}$-module.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow \cancel{\mathbb{Z}/p} \ \mathbb{Z}/p\mathbb{Z} \longrightarrow 0.$$

$$0 \rightarrow \mathbb{Q} \xrightarrow{\cdot p} \mathbb{Q} \longrightarrow 0 \longrightarrow 0.$$

and this is exact.

What about injective modules? One def'n of a module $Q$ being injective is that the (arrow-reversing!) $\mathrm{Hom}(-, Q)$ preserves exactness.

What $\mathbb{Z}$-modules are injective?

Short answer: divisible abelian groups, e.g.

$\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$.

Note that $\mathbb{Z}$ is NOT an injective $\mathbb{Z}$-module.

The lifting property.

"lifting" by analogy.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\exp(ix)"] \\ I \arrow[r] \arrow[ur, dotted, "\text{lift.}"] & S^1 \end{tikzcd}$$

P projective module:

$$\begin{tikzcd} & M \arrow[d] \\ P \arrow[r] \arrow[ur, dotted, "\text{lift.}"] & N \arrow[d] \\ & 0 \end{tikzcd}$$

The lifting map exists by def'n of projective.

Why does this exist for a free module. $F$?

$$\begin{tikzcd} & M \arrow[d, "\phi"] \\ F \arrow[r, "f"] \arrow[ur, dotted, "g"] & N \arrow[d] \\ & 0 \end{tikzcd}$$

Why does $g$ exist? $\phi$ is onto so for each $e_i$ a generator of $F$ $\exists\, m_i \in M$ such that $m_i \in \phi^{-1}(f(e_i))$ $\leftarrow$ nonempty because $\phi$ onto.

Define $g(e_i) = m_i$ and extend by def'n of free

So the plan is: Make it Simpler!

Think through this section for $\mathbb{Z}$-modules.

Earlier we had specific questions about preservation of exactness.

We could ask are "flat" and "injective" the same for $\mathbb{Z}$-modules?

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Not for the quiz: think through the section again with $R = M_{n \times n}(F)$ or $R = \mathbb{C}[G]$ ($G$ a finite group) or $R =$ polyn. ring or $R = \mathbb{Z}[\sqrt{-5}]$ This might involve looking things up!