Math 525 Class 30 November 10

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Let's think about ~~Qz~~ Quiz #4

Proposed example:

$$\begin{tikzcd} & \mathbb{Q}/\mathbb{Z} \arrow[d, "\cdot p"] \\ \mathbb{Q} \arrow[ur, dashed, "\phi"] \arrow[r, "red"] & \mathbb{Q}/\mathbb{Z} \arrow[d] \\ & 0 \end{tikzcd}$$

Does ϕ exist?

We would need $p \cdot \phi(x) = red(x)$ for $x \in \mathbb{Q}$.

try $\phi(x) = \frac{1}{p} \cdot x$

$$\begin{tikzcd} \mathbb{Q} \arrow[r, "red"] \arrow[d, "\cdot \frac{1}{p}"] & \mathbb{Q}/\mathbb{Z} \arrow[d, "\cdot p"] \\ \mathbb{Q} \arrow[r, "red"] & \mathbb{Q}/\mathbb{Z} \end{tikzcd}$$

What other examples could we try?

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Name: Solutions

Consider the exact sequence of abelian groups

$$0 \to \mathbb{Z} \to^{p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$$

where p is a prime.

1. Suppose we apply the functor $- - \otimes \mathbb{Q}/\mathbb{Z}$ to the exact sequence. What is the new sequence of groups and homomorphisms?

$$0 \to \mathbb{Q}/\mathbb{Z} \xrightarrow{\cdot p} \mathbb{Q}/\mathbb{Z} \to 0 \to 0$$

2. With the functor applied as in the previous question, is the sequence still exact? What are the kernels and images at each stage?

There is only one non zero homomorphism left. Its kernel is $\langle \frac{1}{p} \rangle \cong C_p$ and it is onto. The sequence is no longer exact.

3. What is the "lifting property" for projective $\mathbb{Z}$-modules?

$$\begin{tikzcd} & M \arrow[d, "\pi"] \\ P \arrow[ur, dashed, "\exists \tilde{f}"] \arrow[r, "f"'] & N \arrow[d] \\ & 0 \end{tikzcd}$$

If P is projective and $\pi : M \to N$ is onto, then for any $f: P \to N$ $\exists \tilde{f}$ making the diagram commute.

4. Does $\mathbb{Q}$ have this lifting property? Give a proof or a counterexample.

No. $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module because projective $\mathbb{Z}$-modules are free.

OR:

SEE NOTES.

In order to get a working example, we must have a nonzero map $\mathbb{Q} \to N$.

If $\mathbb{Q} \xrightarrow{0} N$, then ~~a is a lift~~ $0 : \mathbb{Q} \to M$ is a well-defined hom making diagram commute.

To get such a nonzero homomorphism N must be "divisible": if $a \in N$, there must be $\frac{1}{n} a \in N$ for ~~n∈~~ $n \neq 0$.

Another way: we have the proof for the equivalent conditions at Prop 30.

We know that. $\mathbb{Q}$ does not satisfy the (1) (3) (4) conditions of prop 30, so it can't satisfy (2).

So we can try to find our counterexample by looking at this proof.

Here ~~are~~ is ~~two~~ an examples

First note that any map $\mathbb{Q} \xrightarrow{\varphi} F$ for any free $\mathbb{Z}$-module F is 0.

This is because F is not divisible.

Any elt. $x \in F$ is $\cancel{\Sigma}$ $a_1 e_1 + \cdots + a_n e_n$ for some subset $\{e_1, \ldots, e_n\}$ of the generators of F and integers a_i.

If $\varphi(1) = x$, then if $n = \max(a_i)$

$$\varphi\left(\frac{1}{2n}\right) = \frac{1}{2n} \cdot x \notin F.$$

Note that there is a map $F \xrightarrow{\pi} \mathbb{Q}$ for an infinitely generated free module F.

Now consider

$$\begin{tikzcd}
 & F \arrow[d, "\pi"] \\
\mathbb{Q} \arrow[ur, dashed, "\widetilde{id}"] \arrow[r, "id"] & \mathbb{Q} \arrow[d] \\
 & 0
\end{tikzcd}$$

There is no map $\widetilde{id}$ making the diagram commute.