Math 525 Class 31 Nov 13

Why is an exact sequence?

It allows for the convenient expression of ideas that would otherwise be cumbersome to address.

Students in Calc 3 study div, grad, curl.

Many ideas can be expressed in the form of an exact sequence.

$$0 \to \mathbb{R} \xrightarrow{i} C^\infty(\mathbb{R}^3, \mathbb{R}) \xrightarrow{\text{grad}} C^\infty(\mathbb{R}^3, \mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\mathbb{R}^3, \mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\mathbb{R}^3, \mathbb{R}) \to 0$$

constant fcns on $\mathbb{R}^3$.

Calc 3: this is an exact sequence.

Simpler version: Calc 1:

$$0 \to \mathbb{R} \xrightarrow{i} C^\infty(\mathbb{R}, \mathbb{R}) \xrightarrow{\text{diff.}} C^\infty(\mathbb{R}, \mathbb{R}) \longrightarrow 0$$

constant fcns.

Note "$0 \longrightarrow A \xrightarrow{f} B$ is an exact seq."

means that f is 1-1.

Also "$C \xrightarrow{g} D \longrightarrow 0$ is exact"

means g is onto.

We saw that given any R-module M,

there is a free R-module F and a surjection.

$$F \xrightarrow{\theta} M$$

Given a free $\mathbb{Z}$-module F, any submodule of this F is also free. (True for any PID.)

So any $\mathbb{Z}$-module A has a "presentation"

$$0 \to F_1 \longrightarrow F_0 \xrightarrow{\theta} A \longrightarrow 0 \quad \text{(an exact seq.)}$$

F_0 = a free module with a map onto A.

$F_1 = \ker \theta$ submodule of F_0, so also free.

Think of this as expressing A in terms of simpler modules. A ~~*~~ 2-term expression might not always be possible.

Example: $\mathbb{F}_2$ is an $\mathbb{F}_2[x]$-module where x acts as 0. $\mathbb{F}_2 \cong \mathbb{F}_2[x]/(x)$

We can try to express $\mathbb{F}_2$ in terms of free $\mathbb{F}_2[x]$-modules as follows:

$$0 \to (x) \longrightarrow \mathbb{F}_2[x] \longrightarrow \mathbb{F}_2 \longrightarrow 0 \quad \text{is exact.}$$

and $(x) \cong \mathbb{F}_2[x]$

BUT: Consider $\mathbb{F}_2[C_2]$ where $\mathbb{F}_2$ is the trivial $\mathbb{F}_2[C_2]$-module. (all elts of C_2 act as 1).

$C_2 = \{e, t\}$

$$\to \mathbb{F}_2[C_2] \xrightarrow{e+t} \mathbb{F}_2[C_2] \xrightarrow{e+t} \mathbb{F}_2[C_2] \to \mathbb{F}_2 \longrightarrow 0$$

This is an exact sequence which does not stop.

(A "free resolution" of $\mathbb{F}_2$ as a $\mathbb{F}_2[C_2]$-module).

(You have something complicated "H_2" - a module which isn't free, and you want to express it in terms of simpler things, say free modules.)

Another reason for the importance of this stuff

"Projective-to-Acyclic" argument.

Suppose $\cdots \to P_n \xrightarrow{d} P_{n-1} \xrightarrow{d} \cdots \to P_0$

is a complex of projectives. $(d^2 = 0)$.

Suppose $\cdots \to A_n \xrightarrow{d} A_{n-1} \xrightarrow{d} \cdots \to A_0$ is

acyclic, i.e. it is an exact seq.

Let's suppose that we have

$$\begin{tikzcd}
P_1 \arrow[d] & A_1 \arrow[d] \\
P_0 \arrow[d] & A_0 \arrow[d] \\
M \arrow[r, "f"] \arrow[d] & N \arrow[d] \\
0 & 0
\end{tikzcd}$$

Then we may lift f to

$f_0 : P_0 \to A_0$

$f_1 : P_1 \to A_1$

etc.

a map of complexes.

Why is this possible? The lifting property of projective modules.

Redraw:

$$\begin{tikzcd}
 & & A_0 \arrow[d] \\
P_0 \arrow[r] \arrow[urr, dashed, "f_0"] & M \arrow[r, "f"] & N \arrow[d] \\
 & & 0
\end{tikzcd}$$

f_0 exists making the diagram commute.

Once we have f_0 we can draw:

$$\begin{tikzcd}
 & & A_1 \arrow[d, "d"] \\
P_1 \arrow[r] \arrow[urr, dashed, "f_1"] & P_0 \arrow[r, "f_0"] & A_0 \arrow[r, Rightarrow] & \mathrm{Im}(d).
\end{tikzcd}$$

If we know $\mathrm{Im}(f_0) \subset \mathrm{Im}(d)$.

then we have from "acyclic" a way to get f_1

The point: the "lifting property" and "exactness" are what we need to make $f: M \to N$ into a map of complexes $f_\bullet: P_\bullet \to A_\bullet$

Another standard argument of alg. top which uses section 10.5.

Suppose

$$
(\ast)\qquad
\begin{tikzcd}
0 \arrow[r] & A_\bullet \arrow[r] \arrow[d, "\theta"] & B_\bullet \arrow[r] \arrow[d] & \left(\overset{B/A}{\cancel{A/B}}\right)_\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_\bullet \arrow[r] & D_\bullet \arrow[r] & \left(\overset{D/C}{\cancel{C/D}}\right)_\bullet \arrow[r] & 0
\end{tikzcd}
$$

A short exact seq. $0 \to A_\bullet \to B_\bullet \to \left(\overset{B}{\cancel{A}}/A\right)_\bullet \to 0$ of complexes gives rise to a long exact seq. in homology.

$$
\to H_{n+1}\left(\overset{B/A}{\cancel{A/B}}\right) \to H_n(A) \to H_n(B) \to H_n\left(\overset{B/A}{\cancel{A/B}}\right) \xrightarrow{\partial} H_{n-1}(A) \to
$$

Compatible maps of complexes [$(\ast)$ commutes.]

give rise to commuting maps in the long exact sequences.

$$
\begin{tikzcd}
\to H_{n+1}\left(\overset{B/A}{\cancel{A/B}}\right) \arrow[r] \arrow[d] & H_n(A) \arrow[r] \arrow[d] & H_n(B) \arrow[r] \arrow[d] & H_n\left(\overset{B/A}{\cancel{A/B}}\right) \arrow[r] \arrow[d] & H_{n-1}\left(\overset{A}{\cancel{A/B}}\right) \arrow[d] \\
\to H_{n+1}\left(\overset{D/C}{\cancel{C/D}}\right) \arrow[r] & H_{n}(C) \arrow[r] & H_n(D) \arrow[r] & H_n\left(\overset{D/C}{\cancel{C/D}}\right) \arrow[r] & H_{n-1}(C)
\end{tikzcd}
$$

If we knew $\theta : A_\bullet \to C_\bullet$ and $\psi : \cancel{A/B} \to \cancel{C/D}$ $B/A \to D/C$

gave isomorphisms in homology.

then the 5-Lemma gives a homology $\cong$

$$H_i(B_\bullet) \longrightarrow H_i(D_\bullet)$$

Last question on previous Quiz:

"$\mathbb{Q}$ is not a projective $\mathbb{Z}$-module"

$$\begin{tikzcd}
 & F \arrow[d, "\pi"] \\
\mathbb{Q} \arrow[ur, dashed, "\theta"] \arrow[r, "id"] & \mathbb{Q} \arrow[d] \\
 & 0
\end{tikzcd}$$

a free module with $\pi : F \to \mathbb{Q}$

any map $\theta : \mathbb{Q} \to F$ is 0.

So it is impossible to find a lift θ making diagram commute.