Math 52T Class 32 Nov 15.

Homework Due Tues. 21^{st}

Quiz Friday Nov ~~16~~ 17^{th} Covers 11.1, 11.2

More Remarks about 10.5

Another important example of an exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 0.$$

(This can be generalized in a variety of ways.)

Exact seq (of right type) $\longrightarrow$ covering space.

Not an exact sequence, but close:

$$\begin{array}{ccc} S^1 & \hookrightarrow & S^3 \\ & & \downarrow \\ & & S^2 \end{array}$$

Notice that the end terms in the exact sequence do NOT determine the middle term.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{(id,\,0)} \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} \xrightarrow{0,\,id} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

Q: Why isn't this a counterexample to the 5-lemma?

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "\cong"] & \mathbb{Z} \arrow[r] \arrow[d, dotted, "\theta"] & \mathbb{Z}/n\mathbb{Z} \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} \arrow[r] & \mathbb{Z}/n\mathbb{Z} \arrow[r] & 0.
\end{tikzcd}$$

A: There is no map. That is, there is no map θ that makes the diagram commute.

Remark: Philosophically similar phenomenon in topology.
$\pi_i(S^2) \cong \pi_i(S^3 \times \mathbb{C}P^\infty)$ for all $i > 0$.
but S^2 and $S^3 \times \mathbb{C}P^\infty$ are not homotopy equiv.
and this is not a counterexample to the Whitehead Theorem beccause there is no map.
$f: S^2 \longrightarrow S^3 \times \mathbb{C}P^\infty$ inducing the isom on π_i.

Another example:

$$\begin{tikzcd}
0 \arrow[r] & C_2 \arrow[r] \arrow[d, equal] & D_8 \arrow[r] & C_2 \times C_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C_2 \arrow[r] & Q_8 \arrow[r] & C_2 \times C_2 \arrow[r] & 0
\end{tikzcd}$$

D_8 and Q_8 are not isomorphic.

Classification of groups that fit into an exact sequence; via 2-cocycles: 17.4.

Def'n of 2-cocycle (p. 824) looks crazy, but in fact you studied this in 2^{nd} grade.

$$\begin{array}{r} {}^{1}12 \\ 9 \\ \hline 21 \end{array}$$

"Carry" is a 2-cocycle for the extension

$$0 \to \mathbb{Z}/10 \to \mathbb{Z}/100 \to \mathbb{Z}/10 \to 0.$$

More about this "Mathematics Made Difficult" By Carl Linderholm (?)

Geometric interpretation of "a projective module"

Let X be a compact space

Vector bundles over X are "the same" as projective modules over ~~$\mathbb{C}$~~ $C(X)$.

Vector bundles are "locally free":

for any $x \in X$, $\exists$ nbd U of x such that the vector bundle $E\restriction_U$ is $\cong U \times \mathbb{R}^n$

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, hook] & E \arrow[d] & & \mathbb{R}^n \arrow[r, hook] & E\restriction_U \arrow[d] \\
& X \arrow[rrr, leftrightarrow] & & & U
\end{tikzcd}$$

Projective modules are "locally free": an R-module P is projective iff for every prime $p \subset R$ the localized module $R_p \otimes_R P$ is free over R_p.

Standard examples of vector bundles that are not globally free. [Globally free means $E \cong X \times \mathbb{R}^n$]

Tangent bundle of S^2.

or: Mobius bundle over S^1.

Remark: Make a vector bundle $\cancel{E}$

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, hook] & E \arrow[d, "p"] \\
& X
\end{tikzcd}$$

into a module over $C(X)$

by considering the module of sections:

$s : X \longrightarrow E$ where we require

s to satisfy $p(s(x)) = x$.

thus $s(x) \in p^{-1}(x) \cong \mathbb{R}^n$ and

can be multiplied by a real number $f(x)$

where $f \in C(X)$.

Flatness is not geometric, at least

not to my knowledge.

Remarks on the 5-lemma

$$\begin{tikzcd}
 & & c & d & \\
A \arrow[r, "\zeta"] \arrow[d, "\alpha"] & B \arrow[r, "\omega"] \arrow[d, "\beta"] & C \arrow[r, "\theta"] \arrow[d, "\gamma"] & D \arrow[r, "\psi"] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\
V \arrow[r, "i"] & W \arrow[r, "j"] & X \arrow[r, "k"] & Y \arrow[r, "l"] & Z \\
 & & x & k(x) &
\end{tikzcd}$$

want to show $\alpha, \beta, \delta, \varepsilon \cong \Rightarrow \gamma$ is $\cong$.

The idea is that the structure makes it

so that every stage of the proof is "forced"

there is only one reasonable thing to do.

Suppose $x \in X$, we want to show that $\exists c \in C$

such that $\gamma(c) = x$. ("onto" part of $\cong$)

Only one thing to do: $k(x) \in Y$.

δ is an isom, so $\exists d \in D$ such that.

$\delta(d) = k(x)$. Now what?

We want to show that $\exists c \in C$ such that

$\theta(c) = d$. Now we check that.

$\ell(\delta(d)) = \varepsilon(\psi(d))$ diag commutes.

$\ell(\delta(d)) = \ell k(x)$ so $\ell(k(x)) = 0$ by exactness.

so $\ell(\delta(d)) = 0$. So: $\cancel{\varepsilon(\psi(d)) =}$

So $\varepsilon(\psi(d)) = 0$.

and ε $\cong$ so so $\psi(d) = 0$.

so $d = \theta(c)$.

Now we ~~have to~~ ~~e~~ $c \in C$ and

$\theta(c) = d$ $\qquad \delta(\theta(c)) = \delta(d) = k(x)$.

while. ~~A~~ $k(\gamma(c)) = k(x)$.

so now $\gamma(c) - x$ satisfies

$$k(\gamma(c) - x) = 0.$$

Thus $\gamma(c) - x = j(\omega)$

β is an isom, so $\omega = \beta(b)$

so we don't have $\gamma(c) = x$ but we can modify c: $\quad c - \omega(b)$

and $$\gamma(c - \omega(b)) = \gamma(c) - \gamma(\omega(b))$$
$$= \gamma(c) - j(\beta(b))$$
$$= \gamma(c) - j(\omega)$$
$$= \gamma(c) - (\gamma(c) - x)$$
$$= x.$$