Math ~~[illegible]~~ 525 Class 33 Nov 17^{th}.

Quiz Solutions next page.

Today: Linear Algebra: there is something to think about.

Let V be the vector space with basis

$$B = \langle e_1, e_2, e_3, \ldots \rangle = \langle e_i \mid i \in \mathbb{N} \rangle$$

What is V^*? $V^* = \mathrm{Dual}(V)$

$$= \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$$

V is the sum $\langle e_1 \rangle \oplus \langle e_2 \rangle \oplus \cdots\cdots$

V^* is an infinite product $\prod_{i=1}^{\infty} \langle e_i \rangle^*$

$V^* = \{ \text{sequences } (a_1, a_2, \ldots, \ldots) \mid a_i \in \mathbb{R} \}$.

This is different from V.

An element of $V = c_1 e_1 + \cdots + c_n e_n$ for some n (a finite sum)

Math 525 — Quiz 15 – November 17

Name: Solutions

1. Let $T\colon V \to V$ be a linear map from a finite-dimensional vector space V to itself, satisfying $T^2 = 0$. Show that $\operatorname{Im}(T) \subset \operatorname{Ker}(T)$.

If $v \in \operatorname{Im}(T)$, $v = T(v')$ so $T(v) = T(T(v'))$

$= T^2(v') = 0$

thus $v \in \ker(T)$.

2. With T as in problem 1, suppose that $\dim V = 5$. What equation is obtained by applying the rank-nullity theorem to T?

$\dim \operatorname{Im}(T) + \dim \ker(T) = \dim V = 5$.

3. With T and V as in problem 2, what is the largest possible value of $\dim \operatorname{Im}(T)$?

$\dim \ker(T) \geq \dim \operatorname{Im}(T)$ and the sum is 5

so the largest possible value is 2.

4. Let $\langle e_1, e_2\rangle$ be the standard ordered basis for $\mathbb{R}^2$ and let $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that $\phi(e_1) = e_2$ and $\phi(e_2) = e_1$. With respect to the ordered basis $\langle e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\rangle$ of $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2$, what is the matrix of $\phi \otimes \phi$?

$$\left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right)$$

must use the finite-dimensionality of something in $T: V \to W$.

What object(s) need to be finite dimensional to make the proof that $\operatorname{rank}(T) = \operatorname{rank}(T^*)$ work, and why?

Maybe this is not obvious?

Other things not obvious: the existence of the determinant.

What should the linear algebra in your mind include?

Generic answer (for any math. subject):

What are the main theorems?

What are the definitions underlying them?

How can the theorems be applied?

How do we prove the theorems?

Let's say that we extend the definition of dimension to be "the cardinality of a basis" for infinite-dim'l vector spaces.

$\dim(V)$ = "countable"

$\dim(V^*)$ = "uncountable"

Exercise: V^* does not have a countable basis.

Fact from linear algebra: row rank = col rank.

or if $T: V \to W$, $\operatorname{rank} T = \operatorname{rank} T^*$

where $\operatorname{rank}(T) = \dim \operatorname{Im}(T)$.

Let $V = \langle e_i \mid i \in \mathbb{N} \rangle$ as above.

Consider $I: V \to V$ identity.

$\operatorname{rank}(I) = \dim(V)$ = "countable"

$\operatorname{rank}(I^*) = \dim(V^*)$ = "uncountable"!

So the fact "row rank = col rank"

In linear algebra:

main theorems: Rank Nullity.
Spectral Theorem

lesser: Existence of det.
Jordan Form
Cayley-Hamilton
Rational Canonical Form.
Smith Normal Form.

In order for Rank-Nullity to make sense, we need ~~a~~ ~~d~~ dimension to be well-defined.

For this we need a fundamental lemma: "Replacement Theorem" aka "Steinitz Exchange Lemma"

Or (matrix approach): Gaussian Elimination and Reduced Row Echelon Form.