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Remarks: In #2 $V \xrightarrow{e} V^{**}$

is 1-1 because we can define an element

$b^* \in V^{**}$ where $b \in B$ is any element

in a basis B of V.

Define $b^*(b) = 1$

$b^*(b') = 0$ where $b' \in B$ $b' \neq b$.

Use this to show nonzero elts of V go to

nonzero elts of V^{**} by extending any

nonzero elt $b \in V$ to a basis B of V.

If B is a basis of V we can exhibit

an element in V^{**} not in the image of e.

~~Define~~ $\cancel{\phi : V^* \to F}$ ~~by~~

[illegible]

Math 525 — Quiz 16 – November 20

Name:______________________________

1. Let V be a vector space and let V^* be the dual of V, so V^{**} is the dual of V^*. There is a natural map $V \to V^{**}$. How do you define this map?

$e : V \to V^{**}$ is defined by $e(v)(f) = f(v)$ for $v \in V$, $f \in V^*$. $e(v) \in \mathrm{Hom}(V^*, F)$.

2. Suppose V is an infinite-dimensional vector space and consider the map $V \to V^{**}$ of problem 1. Is this map one-to-one? Is it onto? (Only yes/no answers are necessary.)

Yes, no.

3. Let A be an $n \times n$ matrix. What is the "i, j cofactor" of A?

$(-1)^{i+j} \cdot \det($ A with ith row and jth col removed$)$

4. What is the "cofactor formula for the inverse" of a matrix $A = (a_{ij})$?

Let b_{ij} = ~~the~~ i,j cofactor of A.

Let $B = (b_{ij})^T$. Then

$$AB = \det(A) \cdot I$$ or

$$\frac{1}{\det(A)} \cdot B = A^{-1}$$

Example: Polynomials:

Basis $\{1, x, x^2, \ldots\}$.

Duals $1^*, x^*, (x^2)^*$, etc.

We can define $\phi = 1^* + x^* + (x^2)^* + \cdots$

This is well-defined on polynomials.

$\phi(g(x)) =$ sum of coeff. of g.

But ϕ is NOT a linear comb of

$1^*, x^*, (x^2)^*$, etc.

Because linear combinations are finite.

Apply this idea to get the element of X^{**}

not in the image.

Figuring out the cofactor formula.

Use the principle: make it ~~sp~~ simpler!

If the problem involves n, or a large number,

try $n=1$, $n=2$, $n=3$ etc.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

You learned this (easily checkable) formula in linear algebra.

This is the cofactor formula!

$$\frac{1}{ad-bc} = \left[\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right]^{-1}$$

What is the cofactor in $\begin{pmatrix} (1,1) & - \\ - & - \end{pmatrix}$ this position?

$$\begin{pmatrix} \cancel{a} & \cancel{b} \\ \cancel{c} & d \end{pmatrix} \quad (-1)^{1+1} \det [d]$$

$$= d.$$

Standard mistake: $\det(-A) = -\det A$.

The determinant is multilinear, not linear.

if A is $n \times n$, $-A$ negates n rows so $\det(-A) = (-1)^n \det A$.

The cofactor formula is used in the proof of the Cayley-Hamilton Theorem: a matrix satisfies its own characteristic polynomial.

Section 11.3 Ex 4.

If V is infinite-dimensional with basis A show that ~~...~~ $A^* = \{v^* \mid v \in A\}$ does NOT span V^*

We will write down an element ϕ of V^* which is not a linear comb of elements of A^*.

Remember $\phi : V \to F$

We need to define $\phi(v)$ for $v \in V$.

Def: $\phi(v) = \phi\left(\sum_{i=1}^{n} c_i e_i\right)$

[v can be expressed in terms of the basis A and $e_i \in A$ are the elements in the expression.]

Now define $\phi(v) = \left(\sum_{i=1}^{n} c_i\right) \in F$

This is well defined because the expression is a finite sum.

Note that $\phi(e) = 1$ for $e \in A$.

But there is no element in $\operatorname{span}(A^*)$ with this property.

m m

Def: $r\varphi(r') = \varphi(r'r)$. $\quad \varphi \in Hom_{\mathbb{Z}}(R, M)$.

Need to show $\quad s(r\varphi) = (sr)\varphi$.

$s\theta$ where $\theta = r\varphi$

$(sr)\varphi(r')$

$= \varphi(r'sr)$.

$s\theta(r') = \theta(r's)$

$= \varphi((r's)\cdot r)$

$= \varphi(r'sr)$

Remarks on 15a

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~~We can define $b^* \in V^*$ for each $b \in B$ as before and then define $\phi \in V^{**}$ to be $\phi(f) = \phi\left(\sum_{i=1}^{n} a_i b_i^*\right)$~~

What is the cofactor in this $\begin{pmatrix} - & (1,2) \\ - & - \end{pmatrix}$ position?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad (-1)^{1+2} \det [c] = -c$$

But the cofactor matrix is transposed!

So the $-c$ goes in the opposite corner.

Thus whatever we think / guess the cofactor formula is can be checked against the 2×2 inversion formula above.

The existence of the determinant is not obvious and the proofs have been refined over time.

Most linear alg. books have a long list of properties of det and very quick proofs.