Math 525 Class 35 Nov 21

Section 10.4 Problem 11, p. 376

$\mathbb{R}^2 = \langle e_1, e_2 \rangle$ Show that $e_1 \otimes e_2 + e_2 \otimes e_1$ cannot be written as a simple tensor $v \otimes w \in \mathbb{R}^2 \otimes \mathbb{R}^2$

The important point: not every element of $\mathbb{R}^2 \otimes \mathbb{R}^2$ (or $M \otimes N$) can be written in the form $v \otimes w$ (or $m \otimes n$)

Every element is a linear combination of such things (simple tensors).

For this problem it is important to know Corollary 19, which gives a basis of $\mathbb{R}^2 \otimes \mathbb{R}^2$.

So a basis of $\mathbb{R}^2 \otimes \mathbb{R}^2$ is

$e_1 \otimes e_1,\ e_1 \otimes e_2,\ e_2 \otimes e_1,\ e_2 \otimes e_2.$

To solve 10.4 #1, write

$$v = a_1 e_1 + a_2 e_2 \qquad w = b_1 e_1 + b_2 e_2$$

and now

$$v \otimes w = a_1 b_1\, e_1 \otimes e_1 + a_1 b_2\, e_1 \otimes e_2 + a_2 b_1\, e_2 \otimes e_1 + a_2 b_2\, e_2 \otimes e_2.$$

If this is equal to $e_2 \otimes e_1 + e_1 \otimes e_2$, using the fact that we have a basis, we can equate coefficients.

Then show: no such v, w.

Correction from last time.

Proof that $e : V \longrightarrow V^{**}$ is injective in general

See next page.

$V \xrightarrow{e} V^{**}$ $\qquad e(v)(f) = f(v)$

$e(v) : V^* \longrightarrow F$

To show: e is 1-1.

if $v \neq 0 \in V$., ETS $e(v) \neq 0 \in V^{**}$

Let B be a basis of V containing v.

(B exists by "Building-Up" Lemma)

Define $f: V \to F$ by $\begin{cases} f(v) = 1 \\ f(b) = 0 & b \neq v \quad b \in B \end{cases}$

Now $f \in V^*$ (enough to define on basis)

$e(v)(f) = f(v) = 1$

Thus $e(v) \neq 0 \in V^{**}$ QED.

More about tensor products:

How do you define an R-module map.

$M \otimes_R N \longrightarrow L$?

A: Use the universal property of the $\otimes_R$.

Define an R-bilinear map.

$$M \times N \longrightarrow L.$$

This extends to $M \otimes_R N \longrightarrow L$

$$\begin{tikzcd}
M \times N \arrow[r] \arrow[dr, "R\text{-bilinear}"'] & M \otimes_R N \arrow[d, dashed, "\exists !"] \\
 & L
\end{tikzcd}$$

In part (c) of 10.3.24 (p. 358)

We assume M is free with basis B and $N_1 \subset M$ is free on $B_1 \subset B$.

Example: $\mathbb{R}^2$ is free on $\langle e_1, e_2 \rangle$.

$\mathbb{R}^1$ is free on $\langle e_1 \rangle$.

What is a basis of $\mathbb{R}^2 / \mathbb{R}^1$? $\langle e_2 + \mathbb{R}^1 \rangle$

So M/N_1 will be free with basis in

1-1 corresp. with $B - B_1$.

The rest of (c) is a general property of free modules: simplest case is

$$\bigcap_{n\geq 1} n\mathbb{Z} = \{0\}.$$

Suppose $m \in \mathbb{Z}$ $m \neq 0$.

Claim: if $|n| > |m|$ $m \notin n\mathbb{Z}$

$= \{\ldots, -n, 0, n, 2n \ldots\}$

So $m \notin \bigcap_{n\geq 1} n\mathbb{Z}$.

Thus $\bigcap_{n\geq 1} n\mathbb{Z} = \{0\}$.

Now generalize to free modules.

10.3.24 (b)

For each $x \in N$ we have

$$x = c_1 b_1 + \cdots + c_{n_x} b_{n_x} \qquad b_i \in B.$$

For each $x \in N$, it is expressible using a finite number of basis elts b_i.

Thus the basis elts necessary to express <u>every</u> $x \in N$ are a countable union of finite sets. which is countable, call this B_1.

Symmetric, Exterior, and Tensor algebras.

The most important case is algebras over a field. and you should know at least the dimension results.

if V is a vector space over F then $V^{\otimes n}$ has dim d^n where $d = \dim V$.

We can assemble $V^{\otimes 1}, V^{\otimes 2}, V^{\otimes 3}$ etc.

$= V, V\otimes V, V\otimes V\otimes V$, etc.

into $T(V) = F \oplus V \oplus V\otimes V \oplus V\otimes V\otimes V \ldots$

of "non-commuting polynomials in a basis of V".

$V = \langle x_1, \ldots, x_d \rangle$

A basis for $V\otimes V$ corresponds to $x_i x_j$

where $i = 1, \ldots, d$ $\quad j = 1, \ldots, d$ so

$x_1 x_2 \neq x_2 x_1$

Multiplication in $T(V)$ is exactly what you'd guess: if $v\otimes w \in V\otimes V$ is a simple tensor and $x \in V$ then

$(v\otimes w)\cdot x = v\otimes w\otimes x \in V\otimes V\otimes V \subset T(V)$

We can define $S(V)$ as a quotient of $T(V)$ by ~~...~~ the ideal generated by all tensors of the form $a\otimes b - b\otimes a$.

Notice that this is a "graded ideal" in $T(V)$.

<u>Example:</u> (not a graded ideal)

$(1+x) \subset \mathbb{Z}[x]$ is NOT a graded ideal:

$$1+x \in (1+x)$$

but $1 \notin (1+x)$

$S(V)$ is the algebra of polynomials in a basis of V.

If $\dim V = n$ then $S^k(V) =$ homog. polyn. of degree k has dim $\binom{n+k-1}{n-1}$

The exterior algebra $\Lambda(V)$

This is the quotient of $T(V)$ by ideal gen by. ~~by~~ all tensors of the form $x \otimes x$.

This gives the usual rules for wedge product.

We write $x \wedge y$ for the image of

$x \otimes y \in T(V)$ in $\Lambda(V)$.

We have $x \wedge y = - y \wedge x$

because

$(x+y) \otimes (x+y)$ is in the ideal.

$$= \underbrace{x \otimes x}_{\text{in ideal}} + x \otimes y + y \otimes x + \underbrace{y \otimes y}_{\text{in ideal}}.$$

So $x \otimes y + y \otimes x$ is in ideal

So $x \wedge y + y \wedge x = 0$ in $\Lambda(V)$

Know that if $\dim V = n$.

$$\dim \Lambda^k(V) = \binom{n}{k}.$$

Also: if $\dim V = n$,

$\dim \Lambda^n(V) = 1$ and this

1-dim'l space is generated by "det".

The book gives lots of examples to show that intuition derived from the field case does NOT work for general rings.

e.g. $T(\mathbb{Q}/\mathbb{Z})$ over $\mathbb{Z}$.

$$\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0.$$

So $$T(\mathbb{Q}/\mathbb{Z}) = \overset{\deg 0}{\mathbb{Z}} \oplus \overset{\deg 1}{\mathbb{Q}/\mathbb{Z}}$$

Example (p. 450)

$\varphi: I \hookrightarrow \mathbb{Z}[x,y] = R.$

$I = (x,y).$

Can form $\Lambda^2(\varphi): \Lambda^2(I) \to \Lambda^2(R)$

where the exterior algebra is formed as

R-modules.

It happens that $\Lambda^2(R) = 0$ since

R is a 1-dim'l free module over itself.

But $\Lambda^2(I) \neq 0$, so φ 1-1 but

$\Lambda^2(\varphi)$ is NOT 1-1.