Math 525 Class 36 November 27.

Over $\mathbb{Z}$, every submodule of a free module is free. This is true even for infinitely generated free modules, but the proof requires the Axiom of choice (or Zorn's Lemma.)

This is also true with a PID in place of $\mathbb{Z}$.

If R is a PID, every submodule of a free module is free. (Even ∞-generated.)

Rotman, *Advanced Modern Algebra*, p. 651.

Note that this implies that every projective module is free!

Super-Fast Summary of Linear Algebra.

The replacement lemma: after reordering, we may replace the elements of a spanning list with elements of a linearly independent list.

In particular, the size of any linearly indep set. is $\leq$ the length of any spanning set.

By def'n a basis is linearly indep and spans.

By def'n dimension = length of any basis.

This is well-defined by the replacement lemma.

$$\text{length}(B_1) \leq \text{length}(B_2) \leq \text{length}(B_1)$$

B_1 lin indep, B_2 spans.

The replacement lemma also shows that any linearly indep. set can be extended to a basis, etc.

Using these ideas we prove the first main thm:

RANK - NULLITY or (~~Fund~~. Fund. Thm of Linear Maps in Axler)

Next Big Theorem : Spectral Theorem.

We consider $T: V \longrightarrow V$ and define eigenvalues and eigenvectors by $Tv = \lambda v$.

Note that we need $T: V \to V$ for this equation to make sense. If we have $T: V \to W$ then $Tv \in W$ and $\lambda v \in V$.

To state the spectral theorem we need the concept of "adjoint". This requires an "inner product". The adjoint of $T: V \to V$ is the operator $T^*: V \to V$ such that

$$\langle T^* v, w \rangle = \langle v, Tw \rangle$$

The Spectral Theorem says that if $T: V \to V$ is self-adjoint $(T = T^*)$ then there is ~~a basis of V~~ an orthonormal basis of ~~F~~ V consisting of eigenvectors for T.

OR: Symmetric matrices can be diagonalized.

(in $\mathbb{R}$ with std. inner product, adjoint = transpose)

Sketch Proof of Spectral Theorem.

Step 1: There exists an eigenvalue.
(More about this later. – it is not obvious.)

Step 2: If T $(=T^*)$ which preserves $U \subset V$ ~~ask~~ (subspace), then T also preserves $U^\perp$, and $T\restriction_{U^\perp}$ is still self-adjoint.

Step 3: Induction on $\dim(V)$.

More about Step 1: We know eigenvalues exist, because they are roots of the characteristic polyn and complex polynomials always have roots.
But this is not enough, because we are working over $\mathbb{R}$ (I'm only talking about the Spectral Thm. for ~~[illegible]~~, IPS FD over $\mathbb{R}$.)
Here we have a special method for finding the largest eigenvalue. Consider the function on $S^{n-1} \subset \mathbb{R}^n$ which is $f(v) = \langle v, Tv \rangle$

if $T = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$ $f(v) = d_1 x^2 + d_2 y^2 + d_3 z^2$.

This is a continuous fcn on on the compact set S^{n-1}. (~~there~~ above: S^2), so it has a maximum.

This maximum is an eigenvector for the largest eigenvalue of T.

Example to keep in mind: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

rotation by 90° counterclockwise in $\mathbb{R}^2$.

Spectral theorem does not apply because this matrix is not symmetric. Eigenvalues are $\pm i$ and the characteristic polynomial is $\lambda^2 + 1$.

Keep this matrix in mind when reading "Rational Canonical Form". Ch. 12 will be the last chapter that we cover.

One more HW: Ch 12 assigned tonight,

due last day of classes (or Sunday Dec 10).

Wed Nov. 29 Quiz 11.5.

Final Exam: (Thurs 14 Dec) Check Univ.

Website, for time & location → CW 314 10:25 Am.

Covers entire course, emphasizes Ch. 10-12.

An important idea from 11.5: Averaging!

Proposition 40; part (1):

$k!$ a unit in R and M = R-module.

The map $\frac{1}{k!}\,\mathrm{Sym}$ gives an isomorphism

between the k^{th} symmetric power $S^k(M)$

and the submodule of symmetric tensors in

$T^k(M)$.

$$\mathrm{Sym}(z) = \sum_{\sigma \in S^k} \sigma z.$$

$$\sigma(m_1 \otimes \cdots \otimes m_k) = m_{\sigma^{-1}(1)} \otimes \cdots \otimes m_{\sigma^{-1}(k)}.$$