Math 525 Class 37 November 29.

Notice that in the quiz, the map T is not actually defined. Does it matter what T is?

Hint: Rank - Nullity Theorem says that if $\dim(V) = 4$ and $\dim(W) = 3$ and $T: V \longrightarrow W$ has rank 2, then there are bases $(x_1, \dots, x_4)$ of V and (y_1, y_2, y_3) of W such that $T(x_1) = y_1$, $T(x_2) = y_2$, $T(x_3) = T(x_4) = 0$.

All ~~the~~ of the constructions Λ^*, S^*, T^* are independent of choice of basis, so whatever result we get from these bases (above) will be true for all T of rank 2.

Another note: There are dimension formulas for questions 1, 3.

But it's possible to take the point of view

S^* = polynomials Λ^* = wedge product

and count by hand:

$V = \langle x_1, x_2, x_3, x_4 \rangle$

$S^2(V) = \langle x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4 \rangle$

has dimension 10.

$S^2(W) = \langle y_1^2, y_2^2, y_3^2, y_1y_2, y_1y_3, y_2y_3 \rangle$

has dim 6.

If V has dim n then $S^d(V)$

has dim $\binom{n+d-1}{n-1} = \binom{n+d-1}{d}$

and $\Lambda^d(V)$ has dim $\binom{n}{d}$.

For Q2, Q4 we can calculate using

the given bases.

So for example

$$T(x_1 \wedge x_2 \wedge x_3) = T(x_1) \wedge T(x_2) \wedge T(x_3)$$

$$= y_1 \wedge y_2 \wedge 0 = 0.$$

and thus $\Lambda^3(T) = 0$.

Another way to think about Q2: T has rank 2, so for any $v_1, v_2, v_3 \in V$, $T(v_1), T(v_2)$, and $T(v_3)$ must be linearly dependent.

Thus $T(v_1) \wedge T(v_2) \wedge T(v_3) = 0$.

For Q4: use the basis specified above and note $T(x_3) = T(x_4) = 0$.

The only nonzero images are $T(x_1^2), T(x_2^2), T(x_1 x_2)$

these images are $y_1^2, y_2^2, y_1 y_2$.

So $S^2(T)$ has rank 3.

Math 525 — Quiz 17 – November 29

Name:______________________________

Let V be a vector space of dimension 4 over $\mathbb{R}$ and let W be a vector space of dimension 3 over $\mathbb{R}$ and let $T\colon V \to W$ be a linear map of rank 2.

1. What are the dimensions of the vector spaces $\Lambda^3(V)$ and $\Lambda^3(W)$?

4 and 1

2. What is the rank of the linear map $\Lambda^3(T)\colon \Lambda^3(V) \to \Lambda^3(W)$?

0

3. What are the dimensions of the vector spaces $S^2(V)$ and $S^2(W)$?

10 and 6

4. What is the rank of the linear map $S^2(T)\colon S^2(V) \to S^2(W)$?

3.

Example: $\mathbb{R}[x] \xrightarrow{\varphi} \mathbb{R}[x]$.

$\varphi(x) = x^2$

$$S_0 \longrightarrow S_0$$
$$S_1 \longrightarrow S_2$$
$$S_2 \longrightarrow S_4$$

According to def'n p. 443, this is a ring homomorphism but NOT a graded ring homomorphism.

Friday Quiz on 12.1

Advice for reading: Do not worry about the proof of the Jordan Canonical Form / Structure theorem for modules over a PID.

Worry instead about the statement (more than 1 version) and the applications.

Important note: the definition of "rank" is NOT obvious.

On p. 460:

the rank of an R-module M over an integral domain R is the maximum number of R-linearly indep elements of M.

It is possible that, for example, M has rank 1 and that there is no single element of M that generates M.

Example from HW: $I = (2, x) \subset \mathbb{Z}[x]$.

I has rank 1 but is not generated by one element.

One justification of the def'n.. p. 471 Ex. 20

R int domain, f.o.f. F, M an R-module

$\operatorname{rank}(M) = \dim_F (F \otimes_R M)$.

In a general ring R, the concept of rank can't be sensibly defined. There are rings R such that

$R \cong R \oplus R$ as R-modules.

There are two forms of the structure theorem for modules over a PID: invariant factor form and primary decomposition form.

Example: $\mathbb{Z}$-module $M \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$.

(invariant factor form)

also $M \cong \left(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\right) \oplus \mathbb{Z}/3\mathbb{Z}$

(primary decomposition)

Know both forms and be able to apply them.

Do not worry about the proofs.