Math 525 Class 18 October 6

HW due date postponed to Monday Oct 9.

New HW due Wed. Oct 11.

Midterm is ~~the~~ Friday Oct 13th.

New HW answers the question:

~~Is~~ Is there a polynomial which is irreducible but for which it is impossible to show with a combination of substitutions $x \longmapsto x+a$ and Eisenstein criteria ~~for~~ for various p, that this polynomial ~~is~~ is irreducible.

Section 9.3 Problem 5 (e).

Part of this problem asks you to show that $I = x\mathbb{Q}[x]$ is NOT finitely generated.

Remember that $\frac{1}{n}x \in I$ for all $n \in \mathbb{Z}$ $(n \neq 0)$

Note that the "degree-0" part of R is just $\mathbb{Z}$, i.e. $R = \mathbb{Z} + x\mathbb{Q}[x] \subset \mathbb{Q}[x]$

R = polyn. in $\mathbb{Q}[x]$ with integer constant term.

All polyn. in I have zero constant term.

Important: $x \in I$ but x does NOT generate I because $\frac{1}{n}x$ is not. the product of x and an element of R since $\frac{1}{n} \notin R$.

Suppose we had some finite collection $f_1, \ldots, f_m$ of polyn. in I which generated I.

Write
$$f_1 = r_{11}x + r_{12}x^2 + \cdots + r_{1d}x^d$$
$$\vdots$$
$$f_m = r_{m1}x + r_{m2}x^2 + \cdots + r_{md}x^d.$$

This is a finite collection of rational numbers
So there is a bound B on the denominators.

Now show that for some large $N > B$ that you can't get $\frac{1}{N}$ x

Must show there is a bound on any possible denominator you can get by combining the f_i, i.e. e.g. $f_1 - f_2 + 3f_3 - 5f_4$.

Section 9.4 Problem 19 (d).

$f(x) = x^p - a \in \mathbb{F}_p[x]$.

$D_x f = px^{p-1} - 0 = 0$.

What is the criterion: f and Df are relatively prime in $\mathbb{F}_p[x]$.

Is this true here?

Q: In showing that $f(x) = x^4 + 1$ is irreducible, we substituted $x+1$ for x and showed $f(x+1)$ is irred. using Eisenstein. Why does this work?

A: $f(x)$ is irred iff $f(x+a)$ is irred.

To see this: note $f(x) = g(x)h(x)$ implies $f(x+a) = g(x+a)h(x+a)$

And vice versa.

Rings you should know about.

Ex $\mathbb{Z}$, $F[x]$, $F[x_1, \ldots, x_n]$,

rings of integers in alg. number fields,

specifically $\mathbb{Z}[i]$ has Euclidean norm.

$\mathbb{Z}[\sqrt{-5}]$ does not have unique fact.

rings of fnctions e.g. $C[0,1]$.

Suggestion: Read Course Summary.

Standard Question: In Chapter 8, we proved that

Every _____ Domain is a

_____ Domain and.

Every _____ Domain is a

_____ Domain

Also: give examples that show that the reverse implication in PID $\Rightarrow$ UFD is not true.

Solution: Using Gauss' Lemma it was shown that $R[x]$ is a UFD iff R is a UFD.

$F[x]$ is a Euclidean Domain, so is a PID and hence a UFD.

So $F[x,y]$ is a UFD.

So $F[x,y]$ is a UFD but NOT a PID because (x,y) is not a principal ideal.

Stuff that will NOT be on the test.

I mentioned two great theorems of alg. number theory.

Finiteness of the ideal class group.
(Measures failure of unique factorization.)

Finite generation of the group of units
(Dirichlet Units Theorem)

[Theorems for Rings of Integers in Alg. Number Fields.]

There are many results about polyn. rings beyond Hilbert's Basis Theorem (Thm 21, p. 316 b.t I can't test any of these, only the basis theorem itself.

The Basis Theorem says:

$R[x)$ is Noetherian if R is Noetherian.

What is the <u>plan</u> of the proof?

Let $I \subset R[x]$ be an ideal.

let L be the set of leading coeff. of polynomials in I. (plus zero)

Then L is an ideal of R.

We may also define L_d = leading terms of polyn. of degree d in I. (+0)

It turns out that:

$$L_1 \subset L_2 \subset \cdots \subset L_d \subset L_{d+1} \subset \cdots \subset L.$$

This is a chain of ideals in R, so finite. Now we only have finitely many polyn. and degrees to work with.

From this we get a set of generators for I.

Category Theory:

In the Category of comm. rings w./ $\underline{1}$

$\mathbb{Z} \longrightarrow \mathbb{Q}$ is an "epimorphism".