Math 525 Class 23 October 23

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Remarks on the midterm:

Recall the hierarchy of learning about a theorem.

(1) Can you state the theorem?

(2) Can you apply the theorem?

(3) Can you prove the theorem?

Example of mistakes in (2):

I asked if $(2, x) \subset \mathbb{Q}[x]$ was a principal ideal.

Intended solution: $\mathbb{Q}[x]$ is an ED, so is a PID so yes, all ideals are principal.

Also possible: $\frac{1}{2} \cdot 2 = 1 \in (2, x)$ so

$$(2, x) = \mathbb{Q}[x] = (1) \text{ is principal.}$$

Math 525 — Quiz 10 – October 23, 2023

Name: Solutions

Consider the polynomial ring $\mathbb{R}[x]$ and the map $\phi\colon \mathbb{R}[x] \to \mathbb{R}[x]$ defined by $\phi(f) = f(x^2)$.

1. Find $\phi(f)$, where $f = 2 - 3x + x^2 \in \mathbb{R}[x]$.

$$\phi(f) = 2 - 3x^2 + x^4$$

2. Is ϕ a ring homomorphism? Why or why not?

Yes. Need to check $\phi(f+g) = \phi(f) + \phi(g)$

and $\phi(f \cdot g) = \phi(f) \cdot \phi(g)$, and $\phi(1) = 1$.

3. Considering $\mathbb{R}[x]$ as a module over itself, is ϕ a homomorphism of $\mathbb{R}[x]$-modules? Why or why not?

No. $\phi(x \cdot 1) = \phi(x) = x^2$

$x\,\phi(1) = x \cdot 1 = x$.

Not equal so not a module homomorphism.

Another example:

We have a result from 9.4 that if $f(x) \in F[x]$ is a polynomial of degree ≤ 3, then f is reducible in $F[x]$ iff it has a root in F.

Erroneous application: let $f(x) = x^4 + x^2 + 1$ in $\mathbb{F}_2[x]$.

Claim: f is irreducible

Pf: $f(0) = 0^4 + 0^2 + 1 = 1$ $f(1) = 1^4 + 1^2 + 1 = 1$

so f has no root and is irreducible by the result above

Mistake: $\deg(f) = 4$ it is not true that $4 \leq 3$.

Also we have $(x^2+x+1)^2 = x^4 + x^2 + 1 \in \mathbb{F}_2[x]$

A little category theory:

diagram p. 354:

$$
\begin{tikzcd}
A \arrow[r, "\text{incl.}"] \arrow[dr, "\varphi"'] & F(A) \arrow[d, "\Phi\ \exists!"] \\
 & M
\end{tikzcd}
$$

The universal property of free module $F(A)$ on the set A.

Heuristic: the free module $F(A)$ is the "vector space" on the basis A over R.

Even though it's easier for most people to think about the heuristic definition, do NOT ignore the universal property.

In particular this will help with 10.4, Tensor Products.

Intuition for tensor products:

Start with $\oplus$: $\mathbb{R}^2 = \langle e_1, e_2 \rangle$

$\mathbb{R}^3 = \langle f_1, f_2, f_3 \rangle$

$\mathbb{R}^2 \oplus \mathbb{R}^3$ is the VS with basis $\langle e_1, e_2, \cancel{e_3} f_1, f_2, f_3 \rangle$

$\oplus$ is to $\otimes$ as $+$ is to $\times$

$\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^3$ is the vector space whose basis is $e_1 \otimes f_1,\ e_2 \otimes f_1,\ e_1 \otimes f_2,\ e_2 \otimes f_2,\ e_1 \otimes f_3,\ e_2 \otimes f_3$.

Notice: $\dim_{\mathbb{R}} (\mathbb{R}^2 \oplus \mathbb{R}^3) = 2 + 3$

$\dim_{\mathbb{R}} (\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^3) = 2 \cdot 3$

Notice also: a way to understand a new concept in modules is to simplify to the special case of vector spaces.

⚠ Note that <u>not</u> every element of $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^3$ is of the form $v \otimes w$ where $v \in \mathbb{R}^2$ and $w \in \mathbb{R}^3$.

Or: not every element of the tensor product is a "simple tensor".

If we write some expression like

$$e_1 \otimes f_1 + e_2 \otimes f_2 + e_1 \otimes f_3$$

The rules of $\otimes$ will not allow us to simplify it to $v \otimes w$ in general.