Math 525 Class 24 October 25

A fundamental argument of category theory:

Objects defined by a universal property are unique (up to isomorphism)

A free module $F(A)$ on a set A is defined by the universal property:

$$\begin{tikzcd} A \arrow[r, "\text{incl}"] \arrow[dr, "\varphi"'] & F(A) \arrow[d, "\Phi"', "\exists!\,\Phi"] \\ & M \end{tikzcd}$$

For any module M and any map ~~of~~ φ of the set A to M there is a unique module map $F(A) \xrightarrow{\Phi} M$

"Uniqversal Property" means that given the setup, the map Φ exists and is unique for <u>any</u> R-module M.

(Q 1): Why is this true for the free module?

If we define the free module to be the set of all (finite) R-linear combinations of elts of A. (This is $\cong$ to R^n if $|A| = n$.)

Then the universal property holds.

Define $\Phi(r_1 a_1 + \cdots + r_m a_m) = r_1 \varphi(a_1) + \cdots + r_m \varphi(a_m)$

Check that this is a map of R-modules.

(That's existence.)

Any map θ making the diagram commute must have $\theta(\mathrm{incl}(a)) = \varphi(a)$ for $a \in A$.

We must therefore have

$$\theta(r_1 a_1 + \cdots + r_m a_m) = r_1 \theta(a_1) + \cdots + r_m \theta(a_m)$$
$$= r_1 \varphi(a_1) + \cdots + r_m \varphi(a_m)$$

So $\theta = \Phi$ and we have uniqueness.

If we instead defined free modules to be "~~the~~ a thing satisfying the universal property" then free modules are unique up to isomorphism.

Suppose $F(A)$ and $\tilde{F}(A)$ are two things satisfying the def'n.

Then

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & \tilde{F}(A) \arrow[d, "\psi"] \\ & F(A) \end{tikzcd}$$

ψ exists by univ prop of $\tilde{F}(A)$.

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & F(A) \arrow[d, "\exists\xi"] \\ & \tilde{F}(A) \end{tikzcd}$$

exists by univ prop. of $F(A)$.

Now consider

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & \tilde{F}(A) \arrow[d, "\xi\psi = \mathrm{id}_{\tilde{F}(A)}"] \\ & \tilde{F}(A) \end{tikzcd}$$

by uniquenes in univ. prop.

Similarly $\psi\xi = id_{F(A)}$. So

we have ~~ξ~~ $F(A)$ and $\tilde{F}(A)$ are isomorphic.

This shows that anything defined by a universal property is unique up to isomorphism.

It does not show that such a thing exists.

Note: it is not a deep argument
it works because the hypothesis "universal property" is so strong.

Q2: In the category of $\mathbb{Z}$-modules

Why doesn't the ~~$\mathbb{Z}$~~ group $\underbrace{\mathbb{Z}/2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/2\mathbb{Z}}_{n \text{ times}}$

satisfy the def'n of the free module on $A = \{a_1, \ldots, a_n\}$?

A2: Either existence or uniqueness must fail.

Uniqueness:

$$\begin{tikzcd}
A = \{a_1, \dots, a_n\} \arrow[r, "\text{incl}"] \arrow[dr, "\varphi"'] & \begin{matrix} \mathbb{Z}/2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/2\mathbb{Z} \\ \langle a_1 \rangle \oplus \quad \oplus \langle a_n \rangle \end{matrix} \arrow[d, dashed, "\Phi"] \\
& M
\end{tikzcd}$$

if Φ exists we must have.

$$\Phi(\text{incl}(a_i)) = \varphi(a_i).$$

and so $\Phi(\cancel{\oplus}\, c_1 a_1 + \cdots + c_n a_n)$

$$= \cancel{\Phi}\, c_1 \Phi(a_1) + \cdots + c_n \Phi(a_n)$$

$$= c_1 \varphi(a_1) + \cdots + c_n \varphi(a_n)$$

This defines Φ on all of $\mathbb{Z}/2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/2\mathbb{Z}$

so Φ is unique.

Existence Fails:

$$\begin{tikzcd}
A = \{a_1\} \arrow[r, "i"] \arrow[dr] & \mathbb{Z}/2\mathbb{Z} = \langle a_1 \rangle \arrow[d, dashed] \\
& \mathbb{Z}
\end{tikzcd}$$

$$a_1 \mapsto 1$$

Q3: Why isn't $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$ a free module on $A = \{a_1, a_2\}$?

A3: Uniqueness fails.

$$\begin{tikzcd}
A=\{a_1,a_2\} \arrow[r, "i"] \arrow[dr, "\varphi"'] & \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \\
 & \langle a_1\rangle \oplus \langle a_2\rangle \oplus \langle x\rangle \arrow[d, "\underline{\Phi}"] \\
 & M.
\end{tikzcd}$$

$\varphi(a_1)$, $\varphi(a_2)$

We can send x to anything in M

so $\underline{\Phi}$ is not unique.

The "universal property" argument shows also that tensor products are unique.

The discussion of tensor products gives several special cases of interest.

One special case is "extension of scalars".

Let's consider $\mathbb{R} \subset \mathbb{C}$. A FDVS / $\mathbb{R}$ is an $\mathbb{R}$-module.

This is $\mathbb{R} \oplus \cdots \oplus \mathbb{R} = \langle e_1, \ldots, e_n \rangle = V$.

We could extend to $\mathbb{C}$ by just letting $\langle e_1, \ldots, e_n \rangle$ be a basis for the FDVS / $\mathbb{C}$.

The tensor product version is $\mathbb{C} \otimes_{\mathbb{R}} V$.

the elements of $\mathbb{C} \otimes_{\mathbb{R}} V$ are sums of "simple tensors" $z \otimes v$ where $z \in \mathbb{C}$ and $v \in V$.

⚠ Warning: Depending on details in the def'n, you may get different types of objects as a result of taking tensor product.

Example: if M is a right R-module and N is a left R-module.

$M \otimes_R N$ is an abelian group.

If R is commutative, then M is also a left R-module and $M \otimes_R N$ is a left R-module, but in general this is not true.

Example: $\mathbb{C}$ is an $\mathbb{R}$-module

$\mathbb{C}$ is also a $\mathbb{C}$-module

$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is a $\mathbb{C}$-module

$\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ is a $\mathbb{C}$-module.

But they are NOT the same.

Quiz on 10.3 Friday. Oct ~~28~~ 27.