Math 525 Class 25 October 27

Quiz solutions next page

New homework is posted. Due Wed. Nov 1

Next quiz 10.4. Oct 30 Mon.

Notes on getting a course:

Suppose you want to have a course on "Quantum Computing": go to Jia Zhao with 4 friends and say "we will register".

Note on HW:

Q: What is the structure of the group $\left(\mathbb{F}_3[x]/(x^2)\right)^\times$

A: $\mathbb{F}_3[x]/(x^2)$ is a comm. ring.

Every elt. can be expressed as $a + bx + I$ where $I = (x^2)$.

Math 525 — Quiz 11 – October 27, 2023

Name: Solutions

1. The inclusion of the rational numbers into the real numbers makes $\mathbb{R}$ a module over $\mathbb{Q}$. Is $\mathbb{R}$ a free $\mathbb{Q}$-module? Why or why not?

Yes. $\mathbb{Q}$ is a field, so a $\mathbb{Q}$-module is a vector space over $\mathbb{Q}$, and all vector spaces are free because AC gives a basis.

2. If V is a finite dimensional vector space over $\mathbb{R}$ and $T\colon V \to V$ is a linear transformation, then V is an $\mathbb{R}[x]$-module, as described in the book. Is V a free $\mathbb{R}[x]$-module? Why or why not?

No. Free $\mathbb{R}[x]$-modules are direct sums of $\mathbb{R}[x]$ with itself, and none of these, except the zero module, ~~are free over $\mathbb{R}$~~ are finite-dimensional over $\mathbb{R}$.

3. Draw the diagram for the "universal property" of the free module $F(A)$ on a set A and explain what the universal property is.

$$\begin{tikzcd} A \arrow[r, "\text{incl}"] \arrow[dr, "\varphi"'] & F(A) \arrow[d, "\Phi"] \\ & M \end{tikzcd}$$

for any module M and any set map $A \xrightarrow{\varphi} M$, $\exists!\ \Phi$ module map $F(A) \to M$.

4. Is $\mathbb{Q}$ a free $\mathbb{Z}$-module on a set consisting of a single element? Why or why not?

No. Existence fails in the universal property.

$$\begin{tikzcd} \{1\} \arrow[r] \arrow[dr, "\varphi"'] & \mathbb{Q} \arrow[d, dotted, "\Phi"] \\ & \mathbb{Z}/2\mathbb{Z} \end{tikzcd}$$

if $\varphi(1) = 1$, there is no Φ.

Thus the ring has order 9, since $a, b \in \mathbb{F}_3$.

When is $a+bx$ invertible mod (x^2)?

In calculus, you learned that

$$\frac{1}{1-x} = 1+x+x^2+\cdots$$

It is also possible to write a series for

$$\frac{1}{a+bx} \quad \text{if} \quad a \neq 0,$$

since $a+bx = a\left(1 - \left(\frac{-b}{a}\right)x\right)$

No worry about convergence since $x^2 = 0$ in $\mathbb{F}_3[x]/(x^2)$.

If $a=0$, $a+bx = bx$ and thus is either a zero-divisor or zero in $\mathbb{F}_3[x]/(x^2)$. $x \cdot bx = bx^2 = 0 \mod (x^2)$

Thus $\left(\mathbb{F}_3[x] / (x^2) \right)^\times$ has order 6.

There are 2 groups of order 6 : C_6 & S_3.

The ring is commutative, so answer $= C_6$.