Math 525 Class 5 Sept. 1

Monday Sept 4. Holiday

Tues. Sept. 5 = Monday

Tues Nov 21 = Friday.

New HW has been assigned!

Question: Let R be a commutative ring with 1. Is there a subset $J \subset R$ such that $RJ \subset J$ but J is not an ideal?

Note that we can have such an example if we drop "commutative". Quiz #4: J = matrices which have zero 1^{st} column, are a left ideal but not a right ideal, and since "ideal" means 2-sided ideal, we have a counterexample.

In the commutative ring with 1 case, ~~[illegible]~~ don't know because in such a ring we can define $2 = 1+1$, $3 = 1+1+1$, -1 = add inv of 1 etc.

Math 525 — Quiz 2 – September 2

Name: Solutions

1. The book states a proposition about polynomials and degrees that includes the statement $\deg(fg) = \deg(f) + \deg(g)$. How is the case of $f = 0$ handled?

The zero polynomial (by def'n of deg) does NOT have a degree.

2. Give an example of a zero divisor in the group ring $\mathbb{Z}[C_3]$ where $C_3 = \langle g \mid g^3 = 1 \rangle$ is the cyclic group with 3 elements.

$$(g^2+g+1)(g-1) = g^3 - 1 = 1 - 1 = 0$$

3. The group ring $\mathbb{R}[Q_8]$ is also a vector space over $\mathbb{R}$. What is its dimension?

$$\dim_{\mathbb{R}}(\mathbb{R}[Q_8]) = |Q_8| = 8$$

4. In the ring $M_2(\mathbb{Z})$, is the set of matrices with first column identically zero a subring?

Yes, in fact it is even an ideal!

$$m \cdot \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} = \begin{pmatrix} | & | \\ mv_1 & mv_2 \\ | & | \end{pmatrix}$$

So if $v_1 = 0$ $\quad m \cdot \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} = \begin{pmatrix} | & | \\ 0 & v_2 \\ | & | \end{pmatrix}$

and it is a (left) ideal.

Note also $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

~~Example~~ Note that $\{0\}$ is an ideal in any ring R.

~~That~~ the ideals in $\mathbb{Z}$ are $n\mathbb{Z}$

that is every ideal of $\mathbb{Z}$ is principal, generated by one element.

This is also true in $F[x]$ where F is a field.

This is not always true: in $\mathbb{R}[x,y]$

the ideal of polynomials with const term 0.

$= (x,y) = \{ f(x,y)\cdot x + g(x,y)\cdot y \mid f,g \in \mathbb{R}[x,y] \}$

Can't be generated by a single element.

Another example: $\mathbb{Z}[\sqrt{-5}]$ has ideals which are not principal.

Note $\mathbb{Z}[\sqrt{-5}] = \mathcal{O}_{\mathbb{Q}(\sqrt{-5})}$ note $-5 \equiv 3 \mod 4$.

This is because $\mathbb{Z}[\sqrt{-5}]$ does not have unique factorization.

$$(1+\sqrt{-5})(1-\sqrt{-5}) = 1^2 - (\sqrt{-5})^2 = 6$$

$$= 2 \cdot 3$$

We can check that $2, 3, 1+\sqrt{-5}, 1-\sqrt{-5}$ cannot be further decomposed.

A ring of integers $\mathcal{O}_K$ in a number field K has unique factorization iff it is a principal ideal domain.

Thus $\exists\ I$ ideal $\subset \mathbb{Z}[\sqrt{-5}]$ which is NOT principal — can we write such a thing down?

Quiz Monday covering 7.4.

An example of $J \subset R$ such that R is a commutative ring with 1 and $RJ \subset J$ but J is not an ideal.

Let J = all integers with ≥ 2 prime factors.

so $6 = 2 \cdot 3 \in J$ $\quad 35 = 5 \cdot 7 \in J$.

Now $6 + 35 = 41$ a prime. $41 \notin J$.

So J is not closed under $+$.