Math 525 Class 6 September 5.

A note on professional communications:

Suppose you have a professor named Jane Smith.

You could refer to her as "Professor Smith" or "Dr. Smith" but NOT "Dr. Jane" or "Professor Jane".

This means the salutation of your letter to her should be "Dear Professor Smith" or "Dear Dr. Smith".

Q: What if her name is Jane Gprxzqbmphski

A: She's got a web page: copy-and paste.

Quiz solutions next page.

Math 525 — Quiz 5 – September 5, 2023

Name: Solutions

1. Fill in the blank: An ideal I in a commutative ring R is prime iff the quotient R/I is ________.

an integral domain

2. Fill in the blank: An ideal I in a commutative ring R is maximal iff the quotient R/I is ________.

a field.

3. Give an example of a ring R and two ideals $I, J \subseteq R$ such that I and J are both maximal but $I \neq J$.

$R = \mathbb{Z}$ $\quad I = 2\mathbb{Z}$ $\quad J = 3\mathbb{Z}$

4. Consider the map $\phi\colon \mathbb{R}[x] \to \mathbb{R}$ from the ring of polynomials to the real numbers defined by $\phi(f) = f(\sqrt{2})$. Is this a ring homomorphism? (Only an answer is necessary.)

YES.

5. Consider the "reduction" map $\phi\colon \mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$.
This reduction map shows that if there is a solution (x, y) to $x^2 + y^2 = 7$ in $\mathbb{Z}$, then there is a solution in $\mathbb{Z}/4\mathbb{Z}$.
Is there a solution (x, y) to $x^2 + y^2 = 7$ in $\mathbb{Z}/4\mathbb{Z}$? How about in $\mathbb{Z}$?

No. We can list all elements of $\mathbb{Z}/4\mathbb{Z}$: $\{0, 1, 2, 3\}$

Thus if $x \in \mathbb{Z}/4\mathbb{Z}$ $x^2 = 0$ or 1 in $\mathbb{Z}/4\mathbb{Z}$.

$7 = 3 \in \mathbb{Z}/4\mathbb{Z}$ so $7 \neq x^2 + y^2$.

AND THEREFORE $7 \neq x^2 + y^2$ in $\mathbb{Z}$.

38 in Section 7.4.

Note that $\frac{2}{6} = \frac{1}{3} \in \mathbb{Q}$ <u>is</u>

a "rational number with an odd denominator".

To be more technically correct the problem might instead say "rational numbers which can be expressed in a form with an odd denominator".

Note also def'n of local ring is in # 37.

17: bar notation for the map $R \to R/I$ is introduced on p. 243.

Example (5) on p. 255

Q: Is it true that if $I \subset R$ is a two-sided maximal ideal then R/I is a field?

A: Not true for non-commutative rings.

Let $R = \mathbb{H}$ and $I = \{0\}$.

Let $R = C(\mathbb{R})$ = continuous functions $\mathbb{R} \to \mathbb{R}$.

I = functions with compact support

$f \in I \iff \exists M$ such that $f(x) = 0$ for $|x| > M$

- I is an ideal of R.
- I is not a prime ideal of R.

(and therefore NOT a maximal ideal).

- Zorn's Lemma says that $I \subset M$ a max. ideal of R.

- Here are some obvious maximal ideals of R.

If $c \in \mathbb{R}$, $M_c = f \in R$ s.t. $f(c) = 0$.

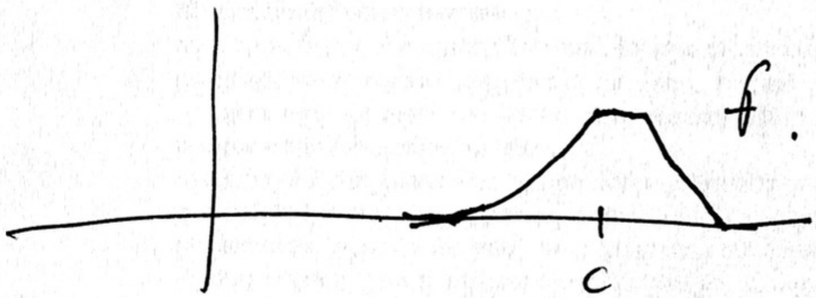
$R \xrightarrow{\phi} \mathbb{R}$ $\quad \phi(f) = f(c)$ "evaluation homomorphism"

$M_c = \ker \phi$ is maximal because the quotient

$R/M_c = \mathbb{R}$ is a field.

The M we constructed from Zorn is NOT any of these. Why?

For each c, we give a function in M but not in M_c.



M is something you "can't write down", analogous to a "nonmeasurable set".

Problem 33 shows that we can't do this if we replace $R = C(\mathbb{R})$ by $C([0,1])$.

Then every maximal ided of $C([0,1])$ is actually of the form M_c for some $c \in [0,1]$.

See 33(a).

Why does the same construction not work here.

We could write $I = \{ f \in C[0,1] \mid \exists \varepsilon,\ f(x) = 0 \text{ if } x < \varepsilon \text{ or } x > 1-\varepsilon \}$

This fails because the M we construct is actually contained in M_0 and M_1.

So the last step fails: we can't prove $M \neq M_c$.

(If ~~...~~ $f(x) = 0$ for $x < \varepsilon$ then $f(0) = 0$ also by continuity.)