Math 525 Class 8 September 8

7.5 "Rings of Fractions"

We already know how to construct $\mathbb{Q}$ from $\mathbb{Z}$. We have rules for adding and multiplying $\frac{a}{b}, \frac{c}{d}$ and rules for equivalence: $\frac{a}{b} = \frac{ah}{bh}$ $(h \neq 0)$

Q: What is required to make this construction work?

A: The denominator can't be zero.

The generalization is we have a set D of "allowed denominators" that satisfies 3 rules: $0 \notin D$, no zero-divisor $\in D$. D closed under mult.

We can have such a set D in any ring R.

Text has examples which address some questions.

e.g. $R = 2\mathbb{Z}$ (a subring of $\mathbb{Z}$).

Let $D = R - \{0\}$.

What is Q? Answer: $Q = \mathbb{Q}$.

$$Q = \left\{ \frac{2a}{2b} \mid a, b \in \mathbb{Z} \quad b \neq 0 \right\}.$$

map $Q \longrightarrow \mathbb{Q}$ $\quad \frac{2a}{2b} \longmapsto \frac{a}{b}$ is isom.

So Q has an identity 1 even though R does not.

Example: $R = \mathbb{Z}[x]$ $\quad D = R - \{0\}$.

$Q = \mathbb{Q}(x)$ = rational functions $\left\{ \frac{f(x)}{g(x)} \mid f, g \in \mathbb{Q}[x] \right\}$.

To see the equivalence, multiply up and clear the denominators.

Notice that question (*) is addressed partially in Corollary 16.

The construction of all elements $\frac{a}{d}$

with rules for $+$, $\cdot$, equivalence,

gives us a ring Q (sometimes $D^{-1}R$).

Q: How, from this construction, do we get $\mathbb{Q}$?

A: $R = \mathbb{Z}$ $D = \mathbb{Z} - \{0\}$.

Q: What else can I get?

A: Problem 38: $R = \mathbb{Z}$ $D =$ odd numbers.

Here $Q = \left\{ \frac{a}{b} \in \mathbb{Q} \;\middle|\; \begin{array}{l} b \text{ is odd} \\ \text{[or there is a representative} \\ \text{with } b \text{ odd.} \end{array} \right\}$.

We seem to have a ring homomorphism.

$$R \xrightarrow{i} Q \qquad i(a) = \frac{ad}{d}.$$

for $d \in D$.

In the case $\mathbb{Z} \xrightarrow{i} \mathbb{Q}$ this is injective.

Is that always so? (☆)

Cor 16: Let R be an integral domain and Q = field of fractions of R. ($D = R - \{0\}$).

if a field F contains a subring R' isom. to R then the subfield of F generated by R' is isom to Q.

In certain circumstances, the map $R \to Q$ must be injective.

Now I think that it should be possible to give an example where $R \longrightarrow Q$ is NOT injective. $Q = D^{-1}R$ for some D, and R need NOT be an integral domain.

But we have this rule that D can't contain 0 or 0-divisors. So maybe we can't try to prove that

$$R \longrightarrow Q \text{ is injective.}^{?}$$

$i(a) = \frac{ad}{d}$ $\quad i(b) = \frac{bd}{d}$ $\quad d \in D.$

assume that $i(a) = i(b)$.

this means $\frac{ad}{d} = \frac{bd}{d}$? What are the rules for equivalence..

See bottom of p. 261.

Need $ad^2 = bd^2$ or $(a-b)d^2 = 0$.

Remember d can't be a zero-divisor in our setting by definition of D.

So the map is injective. because $d^2 \in D$ is not a zero divisor. So $(a-b) = 0$ or $a=b$.

Exercise: 5, p. 265. If F is a field, show that field of fractions of $F[[x]]$ is $F((x))$ = Laurent series.

Show that the field of fractions of $\mathbb{Z}[[x]]$ is NOT $\mathbb{Q}((x))$.

$$F((x)) = \left\{ a_{-N} x^{-N} + a_{-N+1} x^{-N+1} + \cdots + a_0 + a_1 x + \cdots \right\}$$

We want to construct a correspondence

ϕ of Q of $F[[x]]$ with $F((x))$.

$\frac{a(x)}{b(x)} \in Q$ and we want to construct

$c(x) \in F((x))$, which will be the image.

If $b(x) \in F[[x]]$ and $b_0 \neq 0$ then

$b(x)$ has an inverse $f(x) \in F[[x]]$.

So in the case $b_0 \neq 0$ we can write

$$\frac{a(x)}{b(x)} \longmapsto a(x) \cdot f(x) = c(x).$$

What if $b_0 \neq 0$. Then $b(x)$ has inverse $f(x)$.

$$b(x) = x^N (b_0 + b_1 x + \cdots)$$

Then define $\frac{a(x)}{b(x)} \longmapsto x^{-N} a(x) f(x)$.

Well-defined in $F((x))$.

Now part 2: fof of $\mathbb{Z}[[x]]$ is NOT $\mathbb{Q}((x))$. Hint: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Clearly $e^x \in \mathbb{Q}((x))$.

Next step: let $Q =$ fof of $\mathbb{Z}[[x]]$.

Show that $e^x \notin Q$.

There should be an inclusion $Q \hookrightarrow \mathbb{Q}((x))$.

To see that $e^x \notin Q$ try to write

$$\frac{a(x)}{b(x)} = e^x \quad \text{in } \mathbb{Q}((x)).$$

where $a(x), b(x) \in \mathbb{Z}[[x]]$.

Note that $b(x) \in \mathbb{Q}[[x]] \supset \mathbb{Z}[[x]]$

and therefore has an inverse in $\mathbb{Q}((x))$.

So we have $a(x) = e^x b(x)$ in $\mathbb{Q}((x))$.

This implies some specific relations of coeff.

$a(x) = a_0 + a_1 x + \cdots$ $\quad b_0 = b_0 + b_1 x + \cdots$

1	1	$\frac{1}{2}$	$\frac{1}{6}$
b_0	b_0	b_0	$b_0/2$	$b_0/6$
b_1	b_1	b_1	$b_1/2$	$b_1/6$
b_2	b_2	b_2	$b_2/2$	$b_2/6$
b_3	b_3	b_3	$b_3/2$	$b_3/6$

$a_0 = b_0$. $\quad a_1 = b_0 + b_1$ $\quad 2 \mid b_0$?

$a_2 = b_1 + b_2 + b_0/2$ $\quad 3 \mid b_0$?

$a_3 = b_3 + b_2 + b_1/2 + b_0/6$.

Looks like we might need $3 \mid b_0$.

Maybe we can generalize to $p \mid b_0$ for all primes p.

$\Rightarrow$ No such $b_0 \in \mathbb{Z}$.

Quiz 7.6 on Monday.