Math 525 Class 9 September 11

Quiz solutions next page.

General Problem-Solving Heuristic:

"Make it Simpler"

For example, in proof of CRT we first study the simpler case of only 2 ideals.

Often you can understand a proof by induction by careful study of $n = 2, 3$ cases.

Another example, featured prominently in Ch. 8.

"The Extreme Principle".

If you have some measure, it's good to study objects which are extremal w.r.t. that measure.

We have a group of politicians who are to be divided into 2 chambers. Each politician has at most 3 enemies in the group. Show that it is possible to make

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Name: Solutions

1. In Section 7.6 a definition of *comaximal* ideals A and B is given. What does it mean to say that A and B are comaximal?

$$A + B = R$$

2. Does it follow from the definition that if A and B are comaximal, that at least one of A and B is maximal? Give a proof or a counterexample.

No. $A = (4)$ $B = (9)$ $R = \mathbb{Z}$

$1 \cdot 9 - 2 \cdot 4 = 1$ so $A + B = R$

3. What is the order and structure of the group $(\mathbb{Z}/12\mathbb{Z})^\times$?

$$(\mathbb{Z}/12\mathbb{Z})^\times \cong (\mathbb{Z}/4\mathbb{Z})^\times \times (\mathbb{Z}/3\mathbb{Z})^\times$$

$$\cong C_2 \times C_2$$

4. Find all natural numbers x such that $x \equiv 2 \mod 3$, $x \equiv 3 \mod 5$, $x \equiv 4 \mod 7$, and $50 \leq x \leq 105$.

Note $3 \cdot 5 \cdot 7 = 105$ and that by CRT

$\mathbb{Z}/105\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$.

3 equations define the soln mod 3, 5, 7 and hence mod 105. Since $50 \leq x \leq 105$, there is at most one such x. Note $x \equiv 8 \mod 15$ from 1st 2 equations. Now we need only check 53, 68, 83, 98. We see $x = 53 \equiv 4 \mod 7$.

the division in such a way that every politician has at most 1 enemy in his own chamber.

Measurement: sum over all politicians the number of enemies in own. chamber.

Let's consider the division into 2 chambers which minizes this measurement.

Claim: This division has the desired "at most 1 enemy" property.

Proof: Suppose not. Then there is one politician who has 2 enemies in his own chamber. Move him to the other chamber. What happens to the total number of enemies? (It decreases!) That's impossible by the minimality assumption. ◻

Note: The combination of the extreme principle and proof by contradiction is very common.

Relevance to Ring Theory:

In Ch. 8. A "Euclidean Domain" is defined.

This is an integral domain with a Norm N.

Like ~~\$~~ abs value in $\mathbb{Z}$ or deg in $\mathbb{F}[x]$.

If $f, g \in \mathbb{F}[x]$. $f = qg + r$

where q = quotient r = remainder.

such that $\deg(r) < \deg(g)$.

A principal ideal domain is an integral domain in which every ideal is principal, i.e. generated by a single element.

Theorem: Every Euclidean Domain is a P.I.D.

Proof follows the pattern of the proof about politicians that we just did.

If $I \subset R$ (Euclid. Domain), consider the nonzero element $a \in I$ of minimal (nonzero) norm. We claim that $I = (a)$

Suppose NOT. Let $f \in I$, then $f = qa + r$.

where $N(r) < N(a)$. but $f \in I$ & $qa \in I$

So $r \in I$ contradicting minimality of $N(a)$.

Wednesday Sept 13 : Quiz 8.1.