Math 525 Class 10 September 13

Friday Sept 15 : Holiday

Homework will be due Tuesd. 19^{th} instead.

No quiz Monday 18^{th}.

Quiz Wed 20^{th} covering 8.3.

Quiz solutions

Math 525 — Quiz 5 – September 13, 2023

Name: Solutions

1. Complete the following definition: A Euclidean Domain is an integral domain R with a norm N which has the property that for any two elements $a, b \in R$, with $b \neq 0$. there are $q, r \in R$ such that

$a = qb + r$ where $N(r) < N(b)$.

or $r = 0$.

2. Give two examples of Euclidean domains and say what the relevant norms are.

$\mathbb{Z}, |\cdot|$ and $F[x]$, deg.

3. Proposition 1 in Section 8.1 states that every ideal in a Euclidean domain is ...

principal

4. Give an example of an integral domain which is *not* a Euclidean Domain.

Many examples are possible.

$\mathbb{Z}[\sqrt{-5}]$ $\quad$ $\mathbb{Z}[x]$

$F[x,y]$

Example of how to know a theorem

The spectral theorem (for self-adjoint operators on finite dimensional ~~vector~~ inner product spaces.)

Let $T: V \to V$ be a self-adjoint operator on a F.D. inner product space V. Then V has a basis consisting of eigenvectors for T.

[~~Eg.~~ Simpler special case over $\mathbb{R}$:
"Symmetric matrices are Diagonalizable"]

<u>Step 1</u>: There is an eigenvalue (of T)

<u>Step 2</u>: If T preserves $W \subset V$, then T preserves $W^{\perp}$ (and $T\restriction_{W^{\perp}}$ is self-adjoint).

<u>Step 3</u>: Induction on $\dim V$.

Details that did not make the quiz:

"universal side divisor".

Does $\mathbb{Z}$ have a universal side divisor?

Does $F[x]$ ____________________?

Universal Side Divisor means: every element of R can be written $r = qu + b$.

where b is 0 or a unit

[$b \in \tilde{R}$ in book's notation]

and u is the "universal side divisor".

If $R = \mathbb{Z}$ $\tilde{R} = \{-1, 0, 1\} = R^{\times} \cup \{0\}$.

Let $u = 2 \in \mathbb{Z}$. Then every $n \in \mathbb{Z}$ can be expressed as $n = q \cdot 2 + b$ where $b \in \{-1, 0, 1\}$.

So $2 \in \mathbb{Z}$ is a universal side divisor.

What about $F[x]$? (F a field)

$$\widetilde{F[x]} = F[x]^{\times} \cup \{0\} = F.$$

Recall that a "universal side divisor" is required to be an element of $R - \tilde{R}$ (otherwise $u = 1$ work and the concept is stupid.)

So $x \in F[x]$ is a "universal side divisor".

<u>Proposition 5</u>: "This works" in any Euclidean Domain, that is:

In an integral domain R which is not a field, if R is a Euclidean Domain then there are universal side divisors in R.

<u>Proof</u>: Let u be an element $R - \tilde{R}$ of minimal norm. For any $x \in R$, use the def'n of norm to write

$$x = qu + b \quad \text{where } N(b) < N(u)$$
$$\text{or } b = 0.$$

Now if $b = 0$, done.

otherwise $N(b) < N(u) \Rightarrow b \in \tilde{R}^u$.

~~Technic~~ Technical:

$R = \mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$ is NOT

a Euclidean domain.

Plan: show that there is no U.S.D.

Step 1: Show that the units are ~~...~~ ± 1.

So $\tilde{R}^u = \{-1, 0, 1\}$.

How to do this? ~~A~~ Use Norm:

$$N(a + b\sqrt{-19}) = a^2 + 19b^2$$

The norm of a unit must be ± 1.

Step 2: Try to divide some small number by the hypothetical U.S.D. and show that we can't have remainder $0, \pm 1$.