Math 525 Class 13 September 22

We discussed two rings which do <u>NOT</u> satisfy the ACC: $\mathbb{R}[x_1, x_2, \dots]$ polynomials in infinitely many vars.

and $C[0,1]$ continuous fcns $[0,1] \to \mathbb{R}$.

It is possible to give more down-to-earth examples. Ex 5, p. 306

$$R = \mathbb{Z} + x\mathbb{Q}[x] \subset \mathbb{Q}[x]$$

R is a subring which does NOT satisfy the ACC.

R = polynomials in $\mathbb{Q}[x]$ with constant term an integer.

The problem of "factorization not stopping" exists in this ring in the sense that if $n \in \mathbb{Z}$, then $n \cdot \left(\frac{1}{n}x\right) = x \in R$.

Math 525 — Quiz 07 – September 22, 2023

Name: Solutions

1. What are the units in the ring $\mathbb{Z}[x, y]$? ± 1

The units of $R[x]$ are the units of R.
[Prop 1 (2) p. 295]

2. Let R be the polynomial ring $\mathbb{R}[x_1, x_2, \ldots]$, with infinitely many variables. Is

$$z = \sum_{i=1}^{\infty} x_i$$

an element of R? Why or why not? *Hint: how is the polynomial ring in infinitely many variables defined?*

$$\mathbb{R}[x_1, x_2, \ldots] = \bigcup_{n=1}^{\infty} \mathbb{R}[x_1, \ldots, x_n].$$

Thus $z \notin R$ because it is not in any of the $\mathbb{R}[x_1, \ldots, x_n]$

3. Let I be the ideal (7) in the ring $\mathbb{Z}$ and J be the ideal (7) in the ring $\mathbb{Z}[x]$. Are the rings

$$\mathbb{Z}/I[x] \quad \text{and} \quad \mathbb{Z}[x]/J$$

isomorphic?

Yes. See Proposition 2 p. 296.

On the definition of $\mathbb{R}[x_1, x_2, \ldots]$

Note that $\mathbb{R}[x_1] \subset \mathbb{R}[x_1, x_2]$ in a natural way: $2x_1^2 + 3x_1 + 1$ is a polynomial in x_1 and x_2 not involving x_2.

Similarly $\mathbb{R}[x_1, \ldots, x_{n+1}] = \mathbb{R}[x_1, \ldots, x_n][x_{n+1}]$

and $\mathbb{R}[x_1, \ldots, x_n] \subset \mathbb{R}[x_1, \ldots x_{n+1}]$

Analog: $\{1\} \subset \{1,2\} \subset \{1,2,3\} \subset \cdots$

$$\bigcup_{n=1}^{\infty} \{1, 2, \ldots, n\} = \mathbb{N}.$$

Define $z = \sum_{i=1}^{\infty} i$. Is this in $\mathbb{N}$?

No. In fact the equation purporting to define z is not a real definition.

The meaning of $\sum_{i=1}^{\infty} i$ is $\lim_{n \to \infty} \sum_{i=1}^{n} i$

and there is no good concept of limit of $\lim_{n \to \infty} \frac{n(n+1)}{2}$ in $\mathbb{N}$.

More on the definition of polynomials.

$\mathbb{R}[x]$ as a vector space is

$$\begin{array}{ccccccc} \mathbb{R} & \oplus & \mathbb{R} & \oplus & \mathbb{R} & \oplus & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ \langle 1 \rangle & & \langle x \rangle & & \langle x^2 \rangle & & \cdots \end{array}$$

In the polynomial ring $\mathbb{R}[x]$ there is no element

$$\sum_{i=1}^{\infty} x^i$$

A linear combination is by def'n finite.

In Section 9.3 we find Gauss' Lemma which says roughly that factorization in $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$ is (almost) the same, in the sense that (Corollary 6) (p. 304)

If $p(x) \in \mathbb{Z}[x]$ has gcd(coeff) $= 1$, then p is irred. in $\mathbb{Z}[x]$ iff p is irred. in $\mathbb{Q}[x]$

[Works with R a UFD and F = field of fractions]
(in place of $\mathbb{Z}$, $\mathbb{Q}$)

Application of this:
$x^3 - 3x - 1$ is irreducible in $\mathbb{Z}[x]$.

How does this involve Gauss' Lemma?

<u>Proposition</u> (10): A polynomial of degree 2 or 3 over a field F has a root in F iff it is reducible in $F[x]$.

<u>Sketch Pf</u>: If p is reducible it must have a factor of degree 1.

Examples: $(x^2+1)^2$ is reducible in $\mathbb{R}[x]$

but has no root in $\mathbb{R}$.

$(2x-1)(3x-1)$ is reducible in $\mathbb{Z}[x]$

but the roots are $\frac{1}{2}, \frac{1}{3} \notin \mathbb{Z}$.

Proposition 11: Let $p(x) = a_n x^n + \cdots + a_1 x + a_0$

be a polyn. in $\mathbb{Z}[x]$. If $\frac{r}{s} \in \mathbb{Q}$

is a root of p in "lowest terms" [$\gcd(r,s)=1$]

then $r \mid a_0$ and $s \mid a_n$.

Sketch proof: Clear denominators and use divisibility

in $\mathbb{Z}$ plus $\gcd(r,s)=1$.

Back to our application: with prop 10

we see that x^3-3x-1 is irred in $\mathbb{Q}[x]$

iff it has a root in $\mathbb{Q}$.

By Prop. 11, the only possible roots in $\mathbb{Q}$

are ± 1.

But we check $1^3 - 3 \cdot 1 - 1 = -3 \neq 0$

$(-1)^3 - 3 \cdot (-1) - 1 = 1 \neq 0$.

So no root in $\mathbb{Q}$.

So irred in $\mathbb{Q}[x]$

So by Gauss' Lemma, irred in $\mathbb{Z}[x]$.